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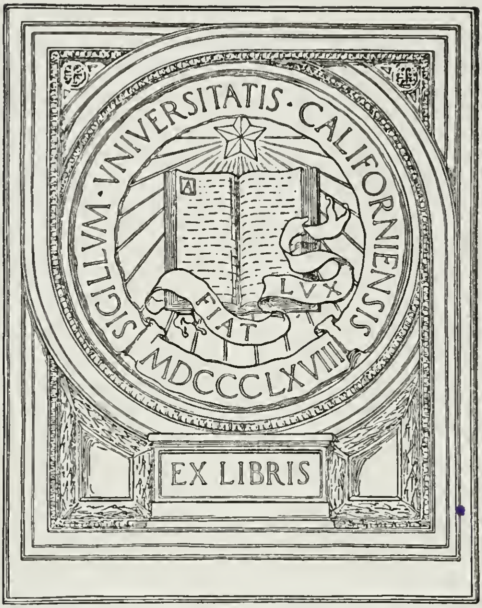

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DESCRIPTIVE GEOMETRY

WITH NUMEROUS PROBLEMS
AND PRACTICAL APPLICATIONS

BY

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WITH TEN FOLDING PLATES

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PREFACE TO FIRST EDITION

THE text books on Descriptive Geometry, with very few exceptions, deal only with first angle projection. But in the best recent practice in mechanical drawing, the third angle is used. The third angle is also commonly employed in perspective. It would seem to be desirable, then, not to confine the student exclusively to the use of any one angle. In this book all four angles are used. Thus the problems become general, and a large variety of constructions may be introduced under each problem.

By having several modifications under each problem, blackboard work can be readily assigned to the members of a class in recitation, and no two students need be given exactly the same work.

If the student is required to be prepared to make the construction for each problem under all possible conditions, a thorough understanding of the problem is necessary.

A large number of carefully arranged problems and some practical applications are given.

The first few problems on the point, line, and plane are fully analyzed. Then follows a list of problems left to the student for solution. In the part of the book devoted to curves and curved surfaces, more problems are analyzed and fewer are left to the student to solve without assistance.

The importance of the study of Descriptive Geometry, both for mental discipline and on account of its industrial utility, is very ably set forth in two quotations that are given at the end of the introductory chapter.

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I desire to take this opportunity of acknowledging my indebtedness to my former teacher, the late Prof. J. M. Silliman, who, for a number of years, gave an excellent course in Descriptive Geometry in Lafayette College.

WM. S. HALL.

EASTON, PA.

PUBLISHER'S NOTE

Previous editions of this book were issued with the "Text" as one volume, and "Illustrations, and Problems for Construction" as a separate volume, but owing to the necessity of constant reference from one volume to the other, we have reconstructed the latter volume so that all material is now included in this one volume edition. There have been no changes in the text, problems or illustrations, in the re-arrangement of the pages and illustrations for this edition. We believe the combined volume will be found more convenient and less cumbersome than the former two volume edition.

D. V. N. CO.

MAY 15TH, 1922.

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DESCRIPTIVE GEOMETRY

CHAPTER I.

FIRST PRINCIPLES

Art. 1. Definition.

1. Descriptive Geometry treats of the representation of geometrical magnitudes by drawings upon plane surfaces and the graphical solution of problems relating to these magnitudes.

Two coördinate planes are commonly employed. In orthographic projections these coördinate planes are perpendicular to each other, and one is usually assumed to be a horizontal plane. The line of intersection of these planes is called the Ground Line. The four diedral angles are distinguished by numbers. In Fig. 1, H represents the horizontal plane, V the vertical plane, and GL is the ground line. H_1 is that part of the horizontal plane in front of the vertical plane, H_2 is that part of H behind V, V_1 is that part of the vertical plane above the horizontal plane, and V_2 is that part of V below H.

Representing the 1st, 2d, 3d, and 4th angles by I, II, III, and IV respectively:

I is between H_1 and V_1.
II " " V_1 " H_2.
III " " H_2 " V_2.
IV " " V_2 " H_1.

Art. 2. Projections of a Point.

2. The planes H and V are called planes of projection. The horizontal projection of a point is the point in which a line from the given point perpendicular to H intersects H. The vertical projection of a point is the point in which a line through the given point perpendicular to V intersects V. Lines projecting points on H are called H-projecting lines, and lines projecting points on V are called V-projecting lines. In Fig. I, a_1^h is the horizontal projection of a_1, and a_1' is its vertical projection. $a_1 a_1^h$ and $a_1 a_1'$ are the projecting lines of a_1. If a plane be passed through these projecting lines it will be perpendicular to H, V and GL. This plane intersects H and V in the lines oa_1^h and oa_1' perpendicular to GL at the same point o. Hence oa_1 is a rectangle, and $oa_1^h = a_1' a_1$ and $oa_1' = a_1^h a_1$; *therefore the distance from the horizontal projection of a point to GL is equal to the distance from the point to the V plane, and the distance from the vertical projection to GL is equal to the distance from the point to the horizontal plane.*

The two projections of a point determine the position of the point in space; for the point a_1 is contained in the line through a_1^h perpendicular to H, and also in the line through a_1' perpendicular to V, therefore it is the point of intersection of these two lines.

a_2^h and a_2' are the projections of a point a_2 in *II*.
a_3^h " a_3' " " " " " a_3 " *III*.
a_4^h " a_4' " " " " " a_4 " *IV*.

If a point be situated in either coördinate plane it will be its own projection on that plane, and the other projection will be in GL.

Art. 3. Revolution of Vertical Plane.

3. In order that both projections of an object may appear on the same plane surface, the V plane is revolved about GL until it coincides with H; thus as seen in Fig. 2, the V plane revolves about GL in the direction shown by the arrows until it coincides with H. In this rotation the vertical projection a_1' describes the arc of a vertical circle, $a_1'a_1^v$, whose center is at o_1 and after revolution, the vertical projection appears at a_1^v. But

$o_1a_1^v = a_1a_1^h$, therefore, the distance from the revolved position of the vertical projection of a point to *GL* is equal to the distance from the point to *H*. And as $o_1a_1^v$ is evidently in the same straight line with $o_1a_1^h$, it follows that after the revolution of *V*, the two projections of a point are in the same perpendicular to *GL*. Likewise, after the revolution of *V*, the two projections of a point a_2 in the second angle are a_2^h and a_2^v, and the two projections of a point a_3 in the third angle are a_3^h and a_3^v.

After the revolution of *V*, the two planes of projection coincide, and taking the plane of the paper on which the drawing is made to represent the coincident planes, the projections of the points shown in Fig. 2 will appear as in Fig. 3. Examination of Figs. 2 and 3 will show that the two projections of points in the four angles are situated with reference to *GL* as follows:

	HOR. PROJ.	VER. PROJ.
1st Ang.	below	above
2d Ang.	above	above
3d Ang.	above	below
4th Ang.	below	below

Thus it will be seen that any two points in the same perpendicular to *GL* may be taken as the projections of a point in space. Hence, if a point in space is to be shown by two orthographic projections, a horizontal line is first drawn for the *GL*. Then an indefinite perpendicular to *GL* is drawn, and distances along this line are laid off equal to the distances from the point in space to the planes of projection above or below *GL* according to the given table.

Art. 4. Projections of a Line.

4. The projection of a line upon a plane is the line which contains the projections of every point of the given line. In Fig. 4, the horizontal projection of the line *ab* is determined by the lines which project its points on *H*, and as these lie in a plane aba^hb^h perpendicular to *H*, the horizontal projection is evidently a^hb^h. Similarly, a plane through the line ab, perpendicular to V, intersects V in a line $a'b'$, which is the vertical projection of ab. After the revolution of V, the vertical projection is shown at a^vb^v.

A plane passed through a line perpendicular to H will be called the H-projecting plane. Likewise a plane through the line perpendicular to V will be called the V-projecting plane.

Either projection of a line is determined in position when the projections of two points of the line on the assumed plane are known. As in Fig. 4, the horizontal projection of the line ab is determined when the horizontal projections of two points a and b are found; then the horizontal projection is the line connecting a^h and b^h.

If through each projection of a line a plane is passed perpendicular to the corresponding plane of projection, each plane will contain the given line in space, and the line is determined by the intersection of the planes. Therefore, any two rectilinear projections can be the projections of one and but one line in space. In Fig. 5, lines M, N, O and L are situated in the first, second, third and fourth angles respectively.

Thus it will be seen that any two lines in a drawing intercepted between two perpendiculars to GL may be taken as the projections of some line in space.

PROB. 1. To draw the projections of a point in any angle, at given distances from the planes of projection.

PROB. 2. To draw the projections of a line in any angle, the ends of the line being at given distances from the planes of projection.

PROB. 3. To draw the projections of lines that are parallel and perpendicular to the planes of projection.

Art. 5. Representation of Planes.

5. If a given plane is not parallel to either of the planes of projection it will intersect them in lines which meet at GL. The intersection of the plane with H is called the *horizontal* *trace* of the plane, and its intersection with V is called the *vertical trace*. As two intersecting lines determine a plane, any plane is determined by its traces. In Fig. 6, the plane MON intersects H in the horizontal trace HP, and intersects V in the vertical trace P'. After the revolution of V, the vertical trace is shown at VP. The two traces intersect at the point O in GL. After the revolution of V the plane MON is represented as in Fig. 7 by the traces HP and VP. The traces of the same plane in the second angle are H_2P and VP, in the third angle H_2P and V_2P, and in the fourth angle, HP and V_2P. In Fig. 7, Q is a plane perpendicular to H and oblique to V, R is a plane perpendicular to V and oblique to H, S is a plane perpendicular to both H and V, and T is a plane parallel to GL.

There are eight different positions which planes have with reference to H and V, as follows:

1. Obl. to H, Obl. to V.
2. Per. to H, Obl. to V.
3. Obl. to H, Per. to V.
4. Per. to H, Per. to V.
5. Per. to H, Par. to V.
6. Par. to H, Per. to V.
7. Par. to GL.
8. Passing through GL.

PROB. 4. To construct the traces of planes in the different diedral angles when the planes make given angles with the planes of projection.

Art. 6. Notation and Conventions.

6. A point in space is designated by a small letter, and its horizontal and vertical projections by the same letter with h or v respectively written above; as in Fig 4, a is a point in space, and a and a^v are its projections.

A line in space is designated by a capital letter, and its horizontal and vertical projections by the same letter with h or v

written above, as in Fig. 4, the line *ab* is denoted by *M*, and its projections by M^h and M^v.

A plane is designated by a capital letter. Its traces are distinguished by prefixing *H* or *V* to this letter. As seen in Figs. 6 and 7, the plane *MON* is denoted by *P*, and its traces by *HP* and *VP*. If the plane is determined by two intersecting lines, as *A* and *B*, it will be denoted by (*A*, *B*). Similarly, (*A*, *b*) denotes the plane determined by the line *A* and the point *b*, and (*a*, *b*, *c*) denotes the plane determined by the three points.

The point at which the eye is situated is called the point of sight. If the lines projecting an object on a plane are perpendicular to that plane the projection is called orthographic. These lines being parallel to each other will intersect at an infinite distance. These projecting lines may be regarded as visual rays, and if the eye is supposed to be placed in a perpendicular line to the plane and at an infinite distance in front of it, the projection will appear from the point of sight to resemble exactly the object it represents. The point of sight is usually taken in the first angle, and objects in this angle are then visible, while objects in the other angles are regarded as concealed by the planes of projection.

Lines that are given or required in any construction are shown as full lines when visible, and as broken lines (lines made of short dashes), when concealed.

Auxiliary lines or lines of construction are shown as broken lines when visible or concealed.

Visible traces of planes, given or required, are shown as full lines. When the traces are invisible, or when the planes are auxiliary, the traces are shown as broken and dotted lines, as H_2P and V_2P in Fig. 7.

The two projections of a point are connected by a dotted line.

A plane or line is called oblique when it is oblique to both planes of projection.

In Descriptive Geometry, lines and surfaces are regarded as indefinite in extent unless limited portions are specified.

When the word *line* is used, a right line will be meant.

When reference is made to two projections or traces, it will be understood that the one first mentioned is the horizontal one, unless the contrary is stated.

Art. 7. Profile Planes.

7. A profile plane is a plane perpendicular to both planes of projection. In Fig. 8, plane P is a profile plane. The projection a_1, of the point a on the profile plane, is the intersection with the plane of a perpendicular through a to the plane. The profile plane is then revolved about its intersection with V until it concides with V, and the profile projection then appears at a_2. After V revolves into H, the profile projection appears at a^p. Denoting the intersection of the profile plane with H by G_1 L_1, the distance from a^p to G_1 L_1 is seen to be equal to the distance from the point a to V.

"The study of Descriptive Geometry possesses an important philosophical peculiarity, quite independent of its high industrial utility. This is the advantage which it so preëminently offers in habituating the mind to consider very complicated geometrical combinations in space, and to follow with precision their continual correspondence with the figures which are actually traced — of thus exercising to the utmost, in the most certain and precise manner, that important faculty of the human mind which is called 'imagination,' and which consists, in its elementary and positive acceptation, in picturing to ourselves, clearly and easily, a large and variable collection of ideal objects, as if they were really before us. . . . While it belongs to the geometry of the ancients by the character of its solutions, on the other hand it approaches the geometry of the moderns by the nature of the questions which compose it. These questions are in fact eminently remarkable for that generality which constitutes the true fundamental character of modern geometry; for the methods used are always conceived as applicable to any figures whatever, the peculiarty of each having only a purely secondary influence."

AUGUST CONTE: *Cours de Philosophie Positive.*

"A mathematical problem may usually be attacked by what is termed in military parlance the method of 'systematic approach'; that is to say, its solution may be gradually felt for, even though the successive steps leading to that solution cannot be clearly foreseen. But a Descriptive Geometry problem must be seen through and through before it can be attempted. The entire scope of its conditions, as well as each step toward its solution, must be grasped by the imagination. It must be 'taken by assault.'"

GEORGE SYDENHAM CLARKE, *Captain Royal Engineers.*

8. TABLE OF ABBREVIATIONS AND CONVENTIONS.

H	Horizontal plane of projection.
V	Vertical plane of projection.
P	Profile plane of projection.
H_1	Horizontal plane of projection in front of V.
V_1	Vertical plane of projection above H.
H_2	Horizontal plane of projection back of V.
V_2	Vertical plane of projection below H.
GL	Ground line.
a, b, c, *etc.*	Given points.
e, f, g, *etc.*	Required points.
k, l, m, n, o, *etc.*	Auxiliary points.
M, N, O, *etc.*	Given lines.
S, T, W, *etc.*	Required lines.
G, H, I, J, *etc.*	Auxiliary lines.
Q, R, L, *etc.*	Given planes.
X, Y, Z, *etc.*	Required planes.
D, H, C, *etc.*	Auxiliary planes.
HP	Horizontal trace of plane P.
VP	Vertical trace of plane P.
I., II., III., IV.	Diedral angles between H and V.
r *and* l	Refer to the angles which projections of lines or traces of planes make with GL. For instance, (45°— r) indicates a line drawn from GL toward the right making an angle of 45° with GL.
a_1, a_2 a_3, *etc.*	Successive positions of point a. Likewise, successive positions of lines will be denoted by numerical subscripts.
′ *and* ″	Placed to the right and above a number indicate feet and inches respectively.
h	Designates a horizontal projection when written above and to the right of another letter (as an exponent).
v	Designates a vertical projection when written above and to the right of another letter (as an exponent).
(G, I)	Plane of the lines G and I.
In	Inaccessible distance.
∞	Infinite distance.
Par.	Parallel.
Per.	Perpendicular.
Hor.	Horizontal.
Ver.	Vertical.
Obl.	Oblique.
M_{ht}	Horizonal trace of line M.
M_{vt}	Vertical trace of line M.

CHAPTER II.

PROBLEMS ON THE POINT, LINE AND PLANE.

Art 8. Traces of a Line.

9. THE horizontal trace of a line is the point in which it intersects H, and the vertical trace of a line is the point in which it intersects V.

PROB. 5. Given the projections of a line to find its traces.

Analysis. To find the horizontal trace. The horizontal trace being in H, is vertically projected in GL. Since every point of a line is vertically projected in the vertical projection of the line, the vertical projection of the horizontal trace is at the intersection of GL and the vertical projection of the line. The horizontal trace of the given line is its own horizontal projection, and it must be in a perpendicular to GL through its vertical projection, and also in the horizontal projection of the given line; therefore, the horizontal projection of the horizontal trace is the intersection of the horizontal projection of the line, and a perpendicular to GL through its vertical projection.

The analysis for finding the vertical trace will be obtained by substituting H for V, V for H, horizontal for vertical, and vertical for horizontal, in the above analysis for the horizontal trace.

In Figs. 9 and 10 the line ab is given.

The horizontal trace c^h is found by continuing the vertical projection $a'b'$ in Fig. 9 until it intersects GL at e^v, and erecting a perpendicular to GL at e^v until it intersects the horizontal projection a^hb^h at e^h. The vertical trace f^v is found by continuing the horizontal projection a^hb^h until it intersects GL at f^h, and erecting a perpendicular to GL at f^h until it intersects the vertical projection at f^v. After the revolution of V into H, the construction is shown as in Fig. 10, $a'b'$ taking the position a^vb^v.

PROB. 6. Given the traces of a line to find its projections.

This problem is the converse of Prob. 5.

Art. 9. Length of a Line.

10. PROB. 7. To find the length of a line joining two given points.

Analysis. As the line connecting the two points contains them, the projections of the line must contain the projections of the points. Hence, the horizontal projection of the given line is the line connecting the horizontal projections of the given points, and the vertical projection of the given line is the line connecting the vertical projections of the given points. If the trapezoid formed by the given line, the H-projecting lines of two of its points and the horizontal projection of the given line, be revolved about the horizontal projection of the given line until it coincides with H, the revolved position of the given line will show the line in its true length.

In Fig. 11, the given points are a and b, the horizontal projection of the line joining the points is a^hb^h, and the vertical projection of the line joining the points is $a'b'$. The trapezoid aa_hb_hb is revolved about a_hb_h until it lies in H, when ab is seen at a_1b_1, which gives the true length of the line. After the revolution of V, the construction is shown in Fig. 12. a^ha_1 is perpendicular to a^hb^h and equal to a_0a^v, and b^hb_1 is perpendicular to a^hb^h and equal to b_0b^v.

If in this analysis horizontal be substituted for vertical, and *vice versa*, and V be put for H, the true length of the line will be found by revolution about the vertical projection into V, as seen in the line a_2b_2.

Art. 10. Points and Lines in Oblique Planes.

11. PROB. 8. If a line is situated in a given oblique plane, and one of its projections is given, to find the other projection.

Analysis. When the horizontal projection is given. If a line lies in a plane, the traces of the line must be in the traces of the plane. Then the horizontal trace of the line, being in the horizontal projection of the line and in the horizontal trace of the plane, must be the intersection of these lines. Likewise, the vertical trace of the line is the intersection of the vertical trace of the plane and the vertical projection of the line.

Each trace of the line is its own projection in the plane in which it lies, and its other projection is in GL. Thus two points of the required projection are found, and joining them gives the required projection. In Fig. 13, the horizontal projection M_h of a line lying in the plane Q is given to find its vertical projection. The intersection of HQ and M^h, M_{ht}^h, is the horizontal trace of the line M, which is vertically projected at M_{ht}^v. The intersection of M^h and GL is M_{vt}^h, the horizontal projection of the vertical trace. The vertical trace must lie in a perpendicular to GL at M_{vt}^h, and also in VQ, therefore at their intersection, M_{vt}^v. M^v, the required vertical projection, connects M_{ht}^v and M_{vt}^v.

PROB. 9. If a point is situated in a given plane and one of its projections is given, to find the other projection.

Analysis. Any line drawn through the given projection of the point in the corresponding plane of projection may be taken as one projection of a line lying in the plane and containing the point. The other projection of this line is found by Prob. 8. Then the required projection of the point will be the intersection of a perpendicular to GL through the given projection of the point with the projection of the line just found.

Art. II. Revolution of Points and Lines.

12. PROB. 10. To find the angle which a given line makes with either of the planes of projection.

Analysis. To find the angle which a line makes with H. Pass a line through one point of the given line perpendicular to H, and revolve the given line about this line until it becomes parallel to *V*. Then the vertical projection of the line after revolution makes an angle with *GL* equal to the required angle.

In Fig. 14, let *M* be the given line, and *L*, perpendicular to *H* through one end of the line, is taken as the axis. In this revolution the point *b* describes the arc of a horizontal circle whose center is in *L*. The horizontal projection of this arc will be an equal arc through b^h having its center at a^h; and as this arc is in a horizontal plane through *b*, its vertical projection will be a straight line through b^v parallel to *GL*.

The given line after revolution is parallel to *V*, hence the horizontal projection M_1^h is parallel to *GL*. The horizontal projection of the end *b* after revolution will be in M_1^h and in $b^hb_1^h$, therefore at their intersection b_1^h. The vertical projection of the end *b* after revolution will be in $b_1^vb^v$, and in a perpendicular to *GL* through b_1^h, therefore at b_1^v. The point *a* being in the axis does not change its position. Therefore b_1^v and a^v are two points in the vertical projection of the line after revolution, giving M_1^v. The angle $ma^vb_1^v$ is the required angle.

PROB. 11. A point is assumed in a given plane, and it is required to find the position of the point after the plane has been revolved about one of its traces into the corresponding plane of projection.

Analysis. When the plane is revolved about its horizontal trace into *H*.

The point will move in a circular arc in a plane perpendicular to the horizontal trace, and after revolution will be found in a line through its horizontal projection perpendicular to the trace. In the revolution the distance from the point to the axis does not change, and will be equal to the hypothenuse of a right triangle, one side of which is equal to the distance from the horizontal projection of the point to the axis, and the other side is equal to the distance from the point in space to *H*.

In Fig. 15, let *Q* be the plane, and *a* a point in the plane. As the point moves in an arc in a plane perpendicular to *HQ*, it is horizontally projected in a^ha_1 perpendicular to *HQ*. A right triangle is constructed with perpendicular sides equal to a^hm

and $a^v n$, and the hypothenuse is the distance from m to the required point a_1.

PROB. 12. To revolve a given point through a given angle about an axis perpendicular to one of the planes of projection.

Analysis. When the axis is perpendicular to V. The point describes in the revolution an arc of a circle lying in plane parallel to V. The vertical projection of the arc is an equal arc having the vertical projection of the axis for its center. The horizontal projection of the arc is a straight line parallel to GL.

In Fig. 16, let a be the point, M the axis, and @ the angle of revolution. With M^v as a center, and $M^v a^v$ as a radius, describe an indefinite arc $a^v a_1^v$, which will be the vertical projection of the arc of revolution. Taking the angle $a^v M^v a_1^v$ equal to @ determines a_1^v, the vertical projection of the point after revolution. The horizontal projection of the point after revolution must be in a line parallel to GL through a^h, and also in a perpendicular to GL through a_1^v, therefore at their intersection a_1^h.

PROB. 13. To revolve a given line through a given angle about an axis perpendicular to one of the planes of a projection.

Analysis. Any two points of the line are revolved about the axis through the given angle as in Prob. 12. The line connecting the revolved positions of these points will be the revolved position of the given line.

PROB. 14. To cause a given line by two successive revolutions to become perpendicular to one of the planes of projection.

Analysis. To bring the line perpendicular to H. The line is first revolved until it is parallel to V, as in Prob. 10. Then an axis, through the stationary point of the line perpendicular to V, is assumed, and the line is revolved about this axis until it becomes perpendicular to H.

In Fig. 17, M is the given line. By Prob. 10, the line is revolved about (a^h, O^v) as an axis until it becomes parallel to V at M_1. Then the line M_1 is revolved about (N^h, a^v) as an axis. The extremity b_1 describes an arc parallel to V, whose vertical projection is an arc passing through b_1^v and having a^v as its center. When the line becomes perpendicular to H, its vertical projection will coincide with O^v, and the vertical projection of the revolved position of b_1 will be at b_2^v. The horizontal projection of the arc described by b_1 will be $b_1^h a^h$, and the horizontal projection of M_1 after revolution will be the point a^h.

Art. 12. Plane Angles.

13. If two lines intersect they have a point in common. The horizontal projection of this common point must be in the horizontal projections of both lines, hence at their intersection. Likewise, the vertical projection of the intersection of the two lines must be the intersection of the vertical projections of the two lines. And as the two projections of a point must be in a perpendicular to *GL*, the line joining the intersections of the corresponding projections of the lines must be in a perpendicular to *GL*. Hence the projections of two intersecting lines may be assumed by taking at pleasure both projections of one line, and the horizontal projection of the other line; then through the point of intersection of the assumed horizontal projections, a line is drawn perpendicular to *GL*, intersecting the vertical projection of the first line, and through this point of intersection any line may be drawn to represent the vertical projection of the second line.

Prob. 15. Given two intersecting lines, to find the traces of the plane passing through them.

Analysis. The horizontal traces of the lines must be in the horizontal trace of the plane, and the vertical traces of the lines must lie in the vertical trace of the plane. Therefore the line connecting the horizontal traces of the given lines will be the horizontal trace of the plane, and the line connecting the vertical traces of the given lines will be the vertical trace of the plane.

In Fig. 18, the lines *M* and *N* are given intersecting at the point *o*. The horizontal traces of *M* and *N* are respectively M_{ht} and N_{ht}; then the line *HX* connecting them will be the required horizontal trace. The vertical traces of *M* and *N* are M_{vt} and N_{vt} respectively, and the connecting line *VX* will be the vertical trace. The two traces intersect on *GL*.

PROB. 16. To construct the traces of a plane containing three given points.

Analysis. By connecting one of the points with the other two by lines, two intersecting lines are obtained which determine the plane of the three given points. Then the traces are found as in Prob. 15.

PROB. 17. To find the angle between two intersecting lines.

Analysis. One of the traces of the plane of the two lines is first found. If the plane be revolved about its horizontal trace into H, or about its vertical trace into V, the angle between the lines in their revolved positions will be the required angle.

In Fig. 19, the given lines are M and N intersecting at o. The horizontal trace of the plane of the two lines is HD. By Prob. 11, the resolved position of o is o_1. The point M_{ht} is in the axis, hence its position does not change; then $M_1{}^h$ is the revolved position of M. Similarly, $N_1{}^h$ is the revolved position of N. Therefore the angle at o_1 is the required angle.

PROB. 18. To construct the projections of a line bisecting the angle between two lines.

Analysis. The angle between the two given lines is found in its revolved position as in Prob. 17. This angle is then bisected by a line, and the projections of this bisecting line are found after the plane of the given lines is revolved back to its original position.

PROB. 19. To construct the projections of a line to make given angles with the planes of projection.

Analysis. The projections of a line of definite length, which makes the given angle with H and is parallel to V, are drawn. Through the projections of one end of the line just assumed, the projections of another line of equal length, making the given angle with V and parallel to H, are drawn. The first line is revolved about a vertical axis passing through its intersection with the second line, and the second line is revolved about an axis perpendicular to V through the same point. When these two rotating lines coincide, a line is found which makes the required angle with each plane of projection.

In Fig. 20, the line M is assumed. M^h being parallel to GL and M^v making the same angle with GL which the required line is to make with H. Then the line N is assumed, N^v being parallel to GL and N^h making the same angle with GL which the required line is to make with V. M^v must be equal to N^h, as lines M and N are to be equal in length. The line M is revolved about L, a vertical line through the point m. In this revolution, the point a describes a horizontal arc, which is horizontally projected in a^hf^h and vertically projected in a^vf. The line N is revolved about O, an axis perpendicular to V through the point m. In this revolution, the end b describes an arc parallel to V, which is horizontally projected in b^hf^h and vertically projected in b^vf^v. When the lines M and N meet, the extremities of the two lines must coincide, and f the point of intersection of the two arcs must be a point of the required line. The point m, being stationary, is another point of the required line. Therefore S, the line connecting m and f, is the required line.

Art. 13. Planes.

14. A horizontal line lying in a plane may be called a *horizontal of the plane.*

The vertical projection of a horizontal line is parallel to GL; hence in Fig. 21, if M is a horizontal of the plane Q, M^v is parallel to GL. The line M must be parallel to HQ as it is in the plane Q and also parallel to H; therefore M^h must be parallel to HQ. The vertical trace of the line M is evidently M_{vt}.

If a line is perpendicular to a plane, each projection of it will be perpendicular to the corresponding trace of the plane. For the H-projecting plane of the line is perpendicular to the given plane and also to H, hence it is perpendicular to the horizontal trace of the given plane; therefore, the horizontal trace of the given plane being perpendicular to the projecting plane is perpendicular to every line in it, and therefore perpendicular to the horizontal projection of the given line. Similarly, it may be proved that the vertical projection of the line is perpendicular to the vertical trace of the plane.

PROB. 20. To find the line of intersection of two planes.

Case I. When the vertical traces and also the horizontal traces intersect within the limits of the drawing.

Analysis. The planes intersect in a line. Since the line of intersection is contained in each plane, its horizontal trace must be in the horizontal trace of each plane, or at the intersection of the horizontal traces of the two planes. Likewise, the vertical trace of the line of intersection must be at the intersection of the vertical traces of the two planes. The traces of the line being two points of the line, by connecting them the line of intersection is obtained.

In Fig. 22, the line of intersection of the given planes Q and R is S. The horizontal trace of S is S_{ht}, the intersection of the horizontal traces of the two planes, and is vertically projected at S^v_{ht} in GL. The vertical trace of S is S_{vt}, the intersection of VQ and VR, and is horizontally projected in GL at S_{vt}^h. The line S connects its two traces.

Case II. When the horizontal or vertical traces do not intersect within the limits of the drawing.

Analysis. Let it be assumed that the horizontal traces do not intersect. If it is assumed that the vertical traces intersect at an accessible distance, one point of the line of intersection is known at once. To obtain another point an auxiliary horizontal plane may be passed cutting both of the given planes. The lines of intersection of this plane with the given planes will meet in a point common to the two planes.

In Fig. 23, the vertical traces of the given planes Q and R meet in the point S_{vt} giving one point of the line of intersection. The auxiliary horizontal plane D cuts from the planes Q and R the horizontals H and L respectively. The intersection f of these horizontals is a point in the required line. The line S, connecting f and S_{vt}, is the required line.

PROB. 21. To find the point in which a given line intersects a given plane.

Analysis. If any plane be passed through the given line it will intersect the given plane in a line containing the required point. The required point being a point of the given line, and also a point of the line of intersection of the two planes, it must be the intersection of the given line with the line of intersection of the two planes.

In Fig. 24, P is the given plane, and M the line. The auxiliary plane passed through the line M is the H-projecting plane of M, its vertical trace being VD. The intersection of the planes D and P is the line whose vertical projection is H^v. The lines M and H intersect in the required point f.

PROB. 22. To find the length of a given line intercepted between two given planes.

PROB. 23. To find the perpendicular distance from a given point to a given plane.

Analysis. The projections of the perpendicular from the point to the plane will pass through the projections of the point. The horizontal projection of the perpendicular will be perpendicular to the horizontal trace of the plane, and its vertical projection will be perpendicular to the vertical trace of the plane. The point in which this perpendicular intersects the plane is found by Prob. 21. The required length is the distance from the given point to the foot of the perpendicular in the plane, and may be found by Prob. 7.

In Fig. 26, the given plane is Q and the given point is a. I^h and I^v are the projections of the perpendicular from a to the plane. This perpendicular intersects the plane at the point o. When the line I is revolved about its horizontal projection into H it appears at I_1, whose length from a_1 to o_1 is the required distance.

PROB. 24. To project a given line on a given plane.

Analysis. From any two points of the line perpendiculars are drawn to the plane, and the line connecting the points in which these perpendiculars intersect the plane is the required projection.

In Fig. 27, Q is the given plane and M is the given line.

From two points of the line as a and b, perpendiculars L and K are drawn to the plane, and the points a_1 and b_1 in which the perpendiculars cut the plane are found. The line M_1 connecting these points is the required line.

PROB. 25. To pass a plane through a given point parallel to a given plane.

Analysis. The corresponding traces of parallel planes are parallel. The projections of a horizontal of the required plane through the given point can be drawn, for its horizontal projection will be parallel to the horizontal trace of the given plane, and its vertical projection will be parallel to GL. The vertical trace of this horizontal will be one point in the vertical trace of the required plane. Through the vertical trace of this horizontal, the vertical trace of the required plane is drawn parallel to the vertical trace of the given plane, and through the point where the required vertical trace meets GL, the horizontal trace is drawn parallel to the given horizontal trace.

In Fig. 28, Q is the given plane and a is the given point. L^h is drawn through a^h parallel to HQ, and L^v is drawn through a^v parallel to GL. L is a horizontal of the required plane, and its vertical trace is L_{vt}. VX is drawn through L_{vt} parallel to VQ, and HX is drawn through d parallel to HQ. X is the required plane.

PROB. 26. To pass a plane through a given point perpendicular to a given line.

Analysis. As the plane is to be perpendicular to the line, the traces of the plane must be perpendicular to the projections of the line. The projections of a horizontal of the required plane through the given point may be drawn; the horizontal projection of this line will be perpendicular to the horizontal projection of the given line, and its vertical projection will be parallel to GL. The vertical trace of the required plane is drawn through the vertical trace of the horizontal of the plane and perpendicular to the vertical projection of the given line. The horizontal trace of the required plane is then drawn perpendicular to the horizontal projection of the given line.

In Fig. 29, M is the given line, and a is the given point. L^h is drawn through a^h perpendicular to M^h, and L^v is drawn through a^v parallel to GL. L is a horizontal of the required plane, and its vertical trace is L_{vt}. VX is drawn through L_{vt} perpendicular to M^v, and HX is drawn through d perpendicular to M^h, and X is the required plane.

Prob. 27. To pass a plane through a given line perpendicular to a given plane.

Analysis. From any point of the given line a perpendicular is drawn to the given plane. This perpendicular and the given line determine a plane perpendicular to the given plane.

In Fig. 30, Q is the given plane, and M is the given line. Through any point of M as o a line L is drawn perpendicular to the plane Q. The plane X of the intersecting lines M and L is the required plane.

Prob. 28. To pass a plane through a given point and parallel to two given lines.

Analysis. Draw a line through the point parallel to each given line. The plane of these lines will be the required plane.

In Fig. 31, M and N are the given lines, and a is the given point. If two lines are parallel, their corresponding projections are parallel. Through a^h, M_1^h and N_1^h are drawn parallel to M^h and N^h respectively; and through a^v, M_1^v and N_1^v are drawn parallel to M^v and N^v respectively. The plane X of the lines M_1 and N_1 is the required plane.

Prob. 29. To pass a plane through a given line parallel to another given line.

Analysis. Through any point of the first line a line is drawn parallel to the second line. The plane of these intersecting lines is the required plane.

Prob. 30. To find the angles which a given plane makes with the planes of projection.

Analysis. To find the angle which a given plane makes with H, an auxiliary plane is passed perpendicular to the horizontal trace of the given plane. This auxiliary plane will intersect the given plane and H, in lines which will make with each other the plane angle which measures the required diedral angle. To find the angle which the given plane makes with *V*, an auxiliary plane perpendicular to the vertical trace is constructed.

In Fig. 32, *Q* is the given plane. *D* is an auxiliary plane, perpendicular to the horizontal trace. *HD* is perpendicular to *HQ*, and as *D* is a vertical plane, *VD* will be perpendicular to *GL*. The vertical projection of the intersection of *D* and *Q* is L^v. The angle which the line *L* makes with its own horizontal projection is the measure of the angle between *Q* and *H*. The true size of this angle may be shown by revolving the *H*-projecting plane of the line *L* about its horizontal trace into *H*, giving the required angle $n^h m^h n'$.

Prob. 31. In Fig. 32, to find the angle which plane *Q* makes with profile plane *P*.

Prob. 32. Given the traces of a plane, to find the true size of the angle between the traces.

Prob. 33. Given one trace of a plane and the angle between the traces, to construct the other trace.

Prob. 34. A corner of a cube is truncated by a plane which makes given angles with two faces. What angle does it make with the third face?

Prob. 35. To construct the traces of a plane that shall make given angles with the two planes of projection.

Analysis. Assume that the required plane makes angles α and β with *H* and *V* respectively. If through any point of *GL* planes be passed perpendicular to the two traces of a given plane, these planes will intersect in a line perpendicular to the given plane. By Prob. 30, each of these auxiliary planes cuts the given plane in a line, which makes with one of the planes of projection the angle which the given plane makes with that plane of projection. If each of these auxiliary planes be revolved into a plane of projection about that trace which is perpendicular to *GL*, the intersection of the auxiliary planes will appear in its true size in each revolved position, and each revolved position of this line will be perpendicular to the revolved position of the intersection of one of the auxiliary planes with the given plane.

In Fig. 33, with any point o of GL as center, and with any length ox as radius, for the perpendicular distance to the required plane, describe a semicircle. Then a line mn, making the angle α with GL, and tangent to this arc will be the revolved position of the intersection of the plane through o perpendicular to the horizontal trace; likewise, a line making the angle β with GL and tangent to the same arc, will be the revolved position of the intersection of the plane through o perpendicular to the vertical trace, giving rs. In the counter-revolution of the line rs, the point r must describe an arc rt, to which the required vertical trace is tangent. But n is a point of the vertical trace; therefore the line nX tangent to the arc rt is the required vertical trace. Similarly, sX tangent to the arc mv described by the point m in the counter-revolution of mn, is the required horizontal trace.

Prob. 36. To locate a point on a given plane at given distances from the two planes of projection.

Prob. 37. To find the shortest distance from a given point to a given line.

Analysis. A perpendicular drawn from the given point to the given line will be the required shortest distance. The length of this perpendicular may be found by passing a plane through the given point and line, and revolving this plane about one of its traces into the corresponding plane of projection. A line drawn through the revolved position of the point perpendicular to the revolved position of the line will be the required distance.

In Fig. 34, M is the given line, and a is the given point. Through a and M a plane is passed whose horizontal trace is HT. When this plane is revolved about HT into H, the point a revolves to a_1 and the line M revolves to M_1. The line S through a_1, perpendicular to M_1, is the required distance.

Prob. 38. To find the angle which a given line makes with a given plane.

Analysis. The angle made by a line with a plane is the same as the angle made by the line with its projection on the plane. If the line be continued until it intersects the given plane, and if another point of the line be projected on the plane, a right triangle will be formed by the given line, its projection, and the line projecting the point. In this right triangle, the angle between the given line and the line projecting one of its points, will be the complement of the required angle. In Fig. 35, *M* is the given line, and *P* is the given plane. Through a point *a* of the line, a perpendicular *H* is drawn to the plane. *HD* is the horizontal trace of the plane of the lines *M* and *H*. Revolving this plane about its horizontal trace into *H*, the lines *M* and *H* are shown at M_1 and H_1, and the angle between M_1 and H_1 is the complement of the required angle.

PROB. 39. To find the angle between two given planes.

Analysis. If a plane be passed perpendicular to the intersection of the given planes, it will be perpendicular to both planes, and will cut lines from these planes that will make with each other an angle which measures the required diedral angle.

In Fig. 36, *R* and *Q* are the given planes, and *H* is their line of intersection. *HD* is the horizontal trace of a plane perpendicular to line *H*. The plane *D* cuts from *R* a line whose horizontal trace is r^h, and a line from *Q*, whose horizontal trace is s^h. If the *H*-projecting plane of the line *H* be revolved about its vertical trace into *V*, H_1 is the revolved position of *H*. If through m_1, the revolved position of *m*, a line m_1n_1 be drawn perpendicular to H_1, it will be the revolved position of a perpendicular to line *H*, lying in plane *D*. If now the plane *D* be revolved about *HD* into *H*, the intersection of the lines cut from *R* and *Q* by *D* will revolve to *o* in H^h, at a distance from *HD* equal to m_1n_1. or^h and os^h are, therefore, the revolved positions of the lines cut from *R* and *Q* by the plane *D*, and the angle between them is the measure of the required angle.

PROB. 40. To find the perpendicular distance between two lines not in the same plane.

Analysis. Through one of the lines a plane is passed parallel to the other line, and the second line is projected on this plane. At the point where this projection intersects the first line, a line is drawn perpendicular to the plane. This line will both intersect and be perpendicular to the given lines, and the required distance is the length along this line between the given lines.

In Fig. 37, *M* and *N* are the given lines. Through any point of *M*, as *c*, a line is passed parallel to *N*; then the plane of this line *L* and the line *M* is parallel to *N*. Through any point of *N* as *r* a perpendicular is dropped to plane *D*, the plane of *M* and *L*. The intersection of this perpendicular *H* with plane *D*, or *s*, is then found. A line *I* drawn through *s* parallel to *N* will be the projection of *N* on plane *D*. At the point *o*, where lines *M* and *I* intersect, the line *J* is erected perpendicular to plane *D*. The line *J* will intersect the line *N* in the point *e*, and *oe* is the required line whose true length may now be found.

Art. 14. Additional Problems for Construction.

15. Prob. 41. A square building is covered by a hip roof. If the hip rafter makes an angle of 30° with *H*, what angle does the roof make with *H*?

Prob. 42. Having given one projection of a line, to construct the other projection; the line to pass through a given point, and to make a given angle with *H* or *V*.

Prob. 43. Fig. 38 represents a cube, with holes drilled at *a* and *b* perpendicular to the faces, with depths of 1″ and $\frac{3}{4}$″ respectively. At what point of the top and at what angle must a hole be started to join the ends of the given holes.

Prob. 44. To revolve a plane through a given angle about an axis perpendicular to one of the planes of projection.

Prob. 45. To find where a line intersects a plane determined by two intersecting lines without finding the traces of the plane of the lines.

Prob. 46. To find the intersection of a plane and a polyedron.

Prob. 47. To find the angle between a diagonal of a cube and one of the edges which it cuts.

Prob. 48. To draw the projections of a regular polygon situated in a given oblique plane.

PROB. 49. To find the projections of a given regular pyramid with its base in a plane that is given.

PROB. 50. Through a point in a line which lies in a given plane, to draw a line which shall lie in the given plane and make a given angle with the given line.

PROB. 51. To find the point of a given plane which is at given distances from the traces.

PROB. 52. Through a given point in a given plane to construct a line which shall lie in the given plane and make a given angle with one of the planes of projection.

PROB. 53. Through a given point not in a given plane to construct a line which shall be parallel to the plane and make a given angle with one of the planes of projection.

PROB. 54. Through a given point in a plane which makes given angles with the planes of projection, to pass a line lying in the plane and making given angles with the two planes of projection.

PROB. 55. Two parallel horizontal shafts are inclined at 30° to a vertical wall. The first shaft is 4′ above, and the second shaft is 5′ below a floor, and the distance between the centers of the shafts is 15′. An 8′ pulley on the first shaft just clears the wall by 6″, and drives a 5′ pulley on the second shaft on the opposite side of the wall by an 18″ belt. Show where the wall and the floor must be cut for the belt. [Scale, $\frac{1}{2}''$ = 1′.]

PROB. 56. To pass a plane through a point and a line and revolve the plane about one of its traces into *H* or *V*, finding the revolved positions of the point and line.

PROB. 57. To pass a plane through two parallel lines and revolve the plane about one of its traces into *H* or *V*, finding the revolved positions of the lines.

PROB. 58. Given one trace of a plane and also the projections of one point in the plane, to construct the other trace.

PROB. 59. Given one trace of a plane and the angle between the two traces, to find the other trace.

PROB. 60. If the traces of a plane are given and the revolved position of a point of the plane when the plane has been revolved about one of its traces into *H* or *V*, to find the projections of the point.

Prob. 61. Given the angle between the traces of a plane, one point in the horizontal trace and the angle which the plane makes with *H*, to construct the traces.

Prob. 62. The lengths of the traces of a plane on *H*, *V* and *P* included between the planes are 2″, $2\frac{1}{2}$″ and 3″ respectively. What angles does this plane make with *H*, *V* and *P*?

Prob. 63. To construct a line through a given point and parallel to two given planes.

Prob. 64. To construct a line which shall be parallel to a given line and shall intersect two other lines not in the same plane.

Prob. 65. To find the point of intersection of three given planes.

Prob. 66. In Fig. 39, a ray of light from *a* is reflected from *V* at *b* to the profile plane *P*, and thence to *H*. Find the points where the reflected ray meets *H* and *P*, and the angles which it makes with *H* and *P*.

Prob. 67. To cause a given oblique plane by two revolutions to become parallel to another given plane.

Prob. 68. A right pyramid has for its base an equilateral triangle of 3″ on a side. What is the altitude of the pyramid if the diedral angles between its faces are 60°?

Prob. 69. To construct a line that shall be parallel to and lie at a given distance from a given plane and that shall also intersect two given lines.

Prob. 70. To find the perpendicular distance between two given parallel planes.

Prob. 71. To construct a line at given distances from two given planes.

Prob. 72. To find the point that is at given distances from three given planes.

Prob. 73. To find a plane passing through a given line and at a certain distance from a given point.

Prob. 74. Through a given point to pass a plane making a certain angle with a given line.

Prob. 75. Through a given point to draw a line that shall be at given distances from two given lines.

Prob. 76. To find the point that is equidistant from four given points.

Prob. 77. To pass a plane through a given line and equidistant from two given points.

Prob. 78. To pass a plane through a given point parallel to a given line and perpendicular to a given plane.

Prob. 79. To erect a perpendicular to a given plane that shall intersect two given lines.

Prob. 80. To find a point in a given line that is at a given distance from a given point.

Prob. 81. Given one projection of a point lying at a given distance from a given line, to find the other projection of the point.

Prob. 82. To cause a given oblique plane by two revolutions to make given angles with H and V, and to find the revolved position of a polygon lying in the given plane.

Prob. 83. To draw a line through a given point that shall intersect a given line, and shall make a given angle with a given plane.

Prob. 84. Given the line of intersection of two planes, the angle between them, and also a point in a trace of one of the planes, to determine the planes.

Prob. 85. To determine the plane bisecting the angle between two given planes.

Prob. 86. To construct a plane through a given line making a given angle with a given plane.

Prob. 87. To find the point of a given line that is equidistant from two given planes.

Prob. 88. To pass a plane through a given line that shall make a given angle with a given plane.

Prob. 89. To determine a plane that shall lie at given distances from three given points.

Prob. 90. To determine a plane perpendicular to one plane of projection and at a given angle with a given line.

PROB. 91. To pass a plane equidistant from two given lines.

PROB. 92. Through a given point to construct a line at a given distance from another point and that shall intersect a given line.

PROB. 93. To find the shortest line that can be drawn parallel to a given plane and terminating in two given lines.

PROB. 94. To determine a plane that shall contain a given line; this line making a given angle with one of the traces of the plane.

CHAPTER III.

CURVES AND TANGENTS TO CURVES.

Art. 15. Generation and Classification of Lines.

16. Every line may be generated by the motion of a point. An *element* of a line is an infinitely short line connecting two consecutive points of the line. The moving point which generates the line is called the *generatrix*. The nature of the line is determined by the law that directs the motion of the generatrix.

Lines may be divided into three classes: 1st, RIGHT LINES, which are generated by a point moving always in the same direction. 2d, PLANE CURVES, which are generated by a point moving in a plane, the direction of its motion continually changing. 3d, CURVES OF DOUBLE CURVATURE, which are generated by a point moving so that no four consecutive points lie in the same plane.

17. A right line is tangent to a curve when it contains two consecutive points of the curve. In Fig. 40, let L be a right line through any two points, as a and b, of the curve C. If the line L be revolved about one of these points as a, the second point of intersection b will gradually approach a. When these points of intersection become consecutive the secant L takes the position of T, and is tangent to the curve at the point a.

Two curves are tangent to each other at a common point, when they may have the same line tangent to each of them at the common point.

If two curves are tangent to each other, their projections on the same plane will be tangent.

18. A normal to a curve at any point is the perpendicular to the tangent to the curve at the point of tangency.

19. Some of the important plane curves are the circle, parabola, ellipse, hyperbola, cycloid, epicycloid, hypocycloid, involute of a circle, conchoid of Nicomedes, logarithmic spiral, spiral of Archimedes, sinusoid, and Schiele's anti-friction curve.

Art. 16. Curves and Tangents.

20. The *parabola* may be generated by a point moving in a plane, so that at any time its distance from a given point called the *focus*, is equal to its distance from a given line called the *directrix*.

PROB. 95. Given the focus and directrix of a parabola, to construct the curve by points, and to draw a tangent to the curve at a point on the curve.

In Fig. 41, let f be the focus and D the directrix. A line drawn through f, perpendicular to D, is called the axis of the curve. To find a point of the curve, draw a perpendicular ap to the axis at any point a, then with f as a center and oa as a radius describe an arc, cutting ap in p. By construction, the point p is equally distant from the point f and the line D, and therefore is by definition a point of the parabola. Similarly, any number of points may be found.

To construct a tangent to the parabola at a point c, a perpendicular to the axis is let fall at c. Then the distance vm from the vertex to the foot of the perpendicular is measured, and an equal distance is laid off on the axis produced from v to t. The required tangent joins the points t and c. (For proof, see "Analytical Geometry.")

21. The *ellipse* may be generated by a point moving in a plane, so that the sum of its distances from two given points, called foci, shall always be equal to a given line.

PROB. 96. Given the foci of an ellipse and the sum of the distances from the foci to a point on the curve, to construct the ellipse by points, and to draw a tangent to the curve at any point.

In Fig. 42, let f and f' be the foci, and vv' the sum of the focal distances to any point of the curve. With either focus as f as center and any distance va, greater than vf and less than vv', as radius, describe an arc. Then with f' as center, and $v'a$ as radius, describe another arc cutting the first one in p. The sum of the distances from p to the foci will be equal to the given line vv', and therefore p is a point of the ellipse. Similarly, any number of points may be found.

To draw a tangent to the ellipse at a point c, a circle circumscribing the ellipse is first drawn, and the perpendicular from c to vv' is extended to intersect the circle at c'. A tangent to the circle at c' is then drawn, cutting vv' produced at t. tc will be the required tangent. (For proof, see "Analytical Geometry.")

The shortest and longest lines which can be inscribed in an ellipse are called conjugate axis and transverse axis respectively. The axes are perpendicular to each other.

PROB. 97. Given the two axes of an ellipse, to construct the curve by concentric circles on the axes, and to draw a tangent from a point without the curve.

In Fig. 43, the axes rr' and vv' bisecting each other at o are given. Then circles are drawn on the axes as diameters. A radius oc is drawn to any point c of the outer circle, and a perpendicular cm is let fall from c to vv'. At the point c', the extremity of radius oc', a line $c'p$ is drawn parallel to vv'. p, the intersection of cm and $c'p$, is a point of the ellipse. Similarly, any number of points may be found.

To draw a tangent to the ellipse from any exterior point, as t. First, with r as a center and the semi-transverse axis as a radius, an arc is drawn which will intersect the transverse axis in the foci f and f'. With either focus as f as center and with a radius equal to the transverse axis, describe an arc ln; then with t as center and its distance from the other focus f' as radius, describe an arc intersecting the arc ln in l and n. By joining the points l and n with f, lines are obtained which intersect the ellipse in x and y, the points of tangency. tx and ty will be required tangents.

(For proof, see "Analytical Geometry.")

22. The *hyperbola* may be generated by a point moving in a plane, so that the difference between its distances from two given points called foci is equal to a given line.

Prob. 98. Given the foci and the transverse axis of an hyperbola to construct the curve by points and to draw a tangent to the curve at any point.

In Fig. 44, let f and f' be the foci and vv' the transverse axis. With either focus as f for a center, and with any radius as fp greater than fv', describe the arc pp'. Then with the other focus f' as center with a radius equal to $fp—vv'$, describe an arc cutting the first arc in p. p is a point of the hyperbola, because the difference between its distances from the foci is equal to a constant distance vv'. Similarly, any number of points in the right branch may be constructed. The left branch is drawn in the same manner.

To draw a tangent to the hyperbola at any point as c, the focal lines fc and $f'c$ are drawn. The tangent ct bisects the angle fcf'. (For proof, see "Analytical Geometry.")

"The hyperbola of most frequent occurrence in the arts is the equilateral, or rectangular hyperbola. It is called equilateral because its axes are equal, and rectangular because its asymptotes are perpendicular. It may be constructed as in Prob. 98 by taking the foci distant from the center of the transverse axis by a length equal to the hypothenuse of a right triangle whose perpendicular sides are the equal semi-axes. In mechanical engineering the curve is commonly constructed as in Fig. 45. OX and OY are perpendicular lines, and a is a given point on the curve. ae is drawn parallel to OX, and through a and any other point as e on ae, perpendiculars af and en are drawn to OX. Then line Oe is drawn intersecting af in l. The line mp through l parallel to OX intersects en in p, which is a point of the required curve. Similarly, any number of points may be found.

23. The cycloid, epicycloid, hypocycloid, and involute are called trochoids.

These curves are proper curves for the teeth in wheel gearing.

24. The common *cycloid* is a plane curve generated by a point in the circumference of a circle as the circle rolls along a right line.

Prob. 99. To construct the common cycloid and to draw a tangent to the curve at any point.

In Fig. 46, let *obd* be the rolling circle, *ox* the right line on which it rolls, and *o* the generating point. To find a point on the cycloid, draw a line *fg* parallel to *ox*, cutting the circle in any point *g*. Then on a line through the center *r*, parallel to *ox*, lay off *rl* equal to arc *og* rectified. Now with *l* as a center, and with a radius equal to the radius of the given circle, draw an arc cutting *fg* in *p*. *p* will be a point of the cycloid, for the point *g* will be in contact with the line *ox* when the center of the rolling circle is at *l*, and the generating point *o* will then have risen just as far as *g* has fallen. Similarly, any number of points on the cycloid may be found.

An approximate rectification of the arc of a circle along a tangent to the arc at one extremity may be readily made as in Fig. 47. Let *r* be the center of a circular arc *oa* which is to be rectified along the tangent *ot*. The chord *oa* is bisected at *c* and produced until $od = oc$. Then with *d* as a center, and *da* as radius, an arc is described cutting *ot* in *b*, and *ob* is the required rectification. If the arc is less than 60°, the approximation is very close, and the error diminishes rapidly as the arc becomes smaller.

To draw a tangent to the cycloid at any point as *a*. When the generating point is at *a*, the center of the rolling circle is at *h*. At this instant every point in the circle is rotating about *m*, which is therefore called the *instantaneous center*; *ma* will then be normal to the curve at *a*, and a perpendicular *at* to this normal will be the required tangent.

The cycloid is known as the curve of quickest descent; i.e., if two points along a right line inclined to the vertical be connected by different curves, a weight will slide from the upper point to the lower one more quickly along the cycloid than along any other curve.

25. An *epicycloid* is a plane curve generated by a point in the circumference of a circle which rolls upon another circle with which it is tangent externally.

PROB. 100. To construct an epicycloid and a tangent to the curve.

In Fig. 48, *R* and *r* are the centers of the fixed and rolling circles respectively, and *o* is the generating point. With *R* as a center, describe an arc *pg*, cutting the rolling circle in any point as *g*. Then the arc *og* is developed on the fixed circle from *o* to *g'*. The radius *Rg'* is continued to intersect the circle described by the center of the rolling circle at *l*. Now with *l* as a center, and with a radius equal to the radius of the rolling circle, describe an arc cutting the arc *pg* in *p*, which will be one point of the epicycloid. Similarly, any number of points may be found.

To draw a tangent to the curve at any point as *p*, a line is drawn from *p* to the instantaneous center *g'*, and the tangent will be perpendicular to this line.

26. The *hypocycloid* is a plane curve which may be generated by a point in the circumference of a circle rolling on the concave arc of another circle to which it is tangent internally.

PROB. 101. To construct the hypocycloid and a tangent to the curve.

The construction of this curve is in close analogy to that of the epicycloid. The construction may be easily understood from Fig. 49, which is lettered to correspond with Fig. 48.

Besides the cycloidal curves already given there are others of less importance. If two circles are tangent internally, and if the larger one rolls upon the smaller, a point in the circumference of the larger circle will generate an *internal* epicycloid or *pericycloid*. If the generating point instead of being on the circumference of the rolling circle is within or without the circumference, other curves are traced. When the circle rolls on a right line, upon another circle in external contact, on another smaller circle in internal contact, or in a larger circle in internal contact, the point within the rolling circle will generate a *prolate trochoid*, *prolate epitrochoid*, *prolate peritrochoid* or *prolate hypotrochoid*, respectively. If the generating point be beyond the circumference of the rolling circle on a radius produced, under conditions as to directrix taken in the same order as just given, it will generate a *curtate trochoid*, *curtate epitrochoid*, *curtate peritrochoid*, or *curtate hypotrochoid*. All of these curves can be readily constructed.

The invention of the cycloid is attributed to Galileo, but it is known that the trochoids were employed by mathematicians, possibly two thousand years before the time of Galileo. Old astronomers explain the motion of the planets on the assumption that they move around a circle whose center moves around another circle.

27. An *involute* of a circle is a plane curve generated by a point in a straight line which rolls on the circumference of a circle. This curve is described by any point in a cord which is kept in tension while it is unwound from a cylindrical spool.

This involute is the proper curve for the working faces of cams in stamp mills. The epicycloid, when the radius of the moving circle is infinity, becomes an involute of a circle.

PROB. 102. To construct an involute of a circle, and a tangent to the curve.

In Fig. 50, r is the center of a given circle, and o is the generating point in a right line tangent to the circle at that point. A point in the involute may be found by drawing a tangent to the circle at any point as b, and rectifying on the tangent the arc from b to o, giving the point p of the involute. Similarly, any number of points may be found.

A tangent to the curve at any point, as p, will be perpendicular to the line pb, which is tangent to the circle at b.

28. The *Conchoid of Nicomedes** is the locus of a point moving in a straight line which rotates about a fixed point on the line called the pole, the generating point always remaining at a constant distance from a given line called the directrix.

PROB. 103. To construct the Conchoid of Nicomedes. In Fig. 51, o is the pole, AB is the directrix, and mn is the constant distance. Any point of the curve may be found by drawing a line as op through the pole, and by taking the distance $rp = mn$, giving the point p. Similarly, any number of points may be found. If the constant distance be laid off above AB, on the line op, another point a of the curve will be established. Thus the curve is seen to have two branches on opposite sides of AB.

This curve may be used in trisecting a plane angle. In Fig. 51, let nop be the given angle. A conchoid is constructed, having one side of the angle as on perpendicular to the directrix AB, and the vertex of the angle is taken as the pole, with the constant distance $mn = 2\ or$. It is assumed that the curve has been so constructed. A line rv is drawn parallel to mn, intersecting the curve at v. Joining o and v gives the angle $nov = \frac{1}{3}\ nop$. Proof: —

$$lv = mn = 2\ or;$$
$$rv = lv \cos lvr = 2\ or \cos nov;$$
$$rv : or :: \sin vor : \sin nov;$$
$$\therefore \sin vor = 2 \sin nov \cos nov$$
$$= \sin 2\ nov.$$
$$\text{Hence, } \angle vor = 2 \angle nov.$$

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* Probably the three most famous problems of the ancients were the squaring of the circle, the duplication of the cube, and the trisection of an angle. One of the most interesting curves invented by these early mathematicians for the purpose of trisecting an angle was the conchoid.

29. The *logarithmic spiral* is a plane curve whose construction depends on this property: Any radius vector which bisects the angle between two other radii is a mean proportional between them.

PROB. 104. To construct the logarithmic spiral.

In Fig. 52, r is the pole, and two points, as a and e, on opposite sides of the pole are assumed to be on the curve. Another point as c may then be found by drawing rc perpendicular to ae, intersecting a circle on ae as diameter in the point c, $\therefore rc^2 = ar \times re$ as required. If cg is drawn perpendicular to ce until it intersects cr produced, the intersection g will be another point of the curve. Similarly, any number of points may be found. This spiral is frequently called the *equiangular* spiral, from the fact that the angle between any radius vector and the tangent to the curve at its extremity is constant.

The logarithmic spiral is used in designing speed-pulleys, and for the outlines of spiral cams and lobed wheels. The logarithmic spiral may be constructed more readily as follows: Any two lines are drawn intersecting each other as ON and OM in Fig. 53. Near the vertex any line as ax is drawn intersecting ON and OM. Then with O as a center and Ox as a radius, the arc xb is drawn. Then through the point b, where the arc intersects OM, the line by is drawn parallel to ax. And again with O as a center, and Oy as a radius, the arc yc is described and so on, until as many points a, b, c, d, etc., are obtained as may be desired. It may be readily proved that any one of these distances as Of is a mean proportional between the distances Og and Oe between which it lies. Radii of a circle running to points of equal division on the circumference are now constructed, and the distances, Oa, Ob, Oc, etc., are laid off on successive radii. The curve connecting the points thus obtained will be the logarithmic spiral.

30. The *spiral of Archimedes* is a plane curve which may be generated by a point moving uniformly along a right line, which rotates uniformly about one of its points. The moving line is called the *radius vector*, and the stationary point is the *pole*.

PROB. 105. To construct the spiral of Archimedes.

Take any point as r for a center, as in Fig. 54, and describe a circle with any radius, as ra. Then divide this circle into any number of equal parts as $ab = bc = cd$, etc. Assume any point on one of the radii as p_1 on ra as one point of the required curve. Now draw the radii rb, rc, etc., and on rb take $rp_2 = 2\ rp_1$, and on rc take $rp_3 = 3\ rp_1$, etc. Thus points, p_1, p_2, p_3, etc., points on the curve, may be constructed.

To draw a tangent to the spiral of Archimedes at any point of the curve, as p_3, any two radii vectors as rx and ry are drawn, making an angle with each other equal to a radian, 57° 29, intersecting the spiral in the points x and y. Then a circle is drawn with its center at r, and having a radius equal to the difference between ry and rx. A line through p_3 tangent to this circle will be normal to the curve at p_3.

31. The *sinusoid* is a plane curve which may be generated by a point moving in a plane, so that in any position its distance from a given right line is equal to the sine of an arc, which is proportional to the distance the point has passed in a direction parallel to the fixed line.

PROB. 106. To construct the sinusoid.

The curve may be constructed by a system of rectangular coordinates, taking the abscissas proportional to arcs of a circle, while the corresponding ordinates are sines of the arcs.

The sinusoid is the vertical projection of a helix, which construction is given in the next problem.

This curve expresses the law of vibration of perfectly elastic solids, and the law of the vibratory movement of the pendulum.

PROB. 107. To construct Schiele's anti-friction curve.

In Fig. 55, ab is the radius of the shaft and bc its axis. The axis bc is divided into equal small divisions at the points 1, 2, 3, etc. The line 1-a is drawn. Then with point 2 as a center, and a radius equal to ab, an arc is described cutting the line 1-a in d, one point of the curve. Similarly, any number of points of the required curve ae are constructed.

32. The ordinary *helix* is a curve generated by a point moving on the surface of a cylinder of revolution, so as to cut all the elements at a constant angle. The moving point has uniform motion around the axis of the cylinder combined with uniform motion parallel to it.

PROB. 108. To construct the projections of a helix and a tangent to the curve at any point, and also to develop the curve on a plane surface.

If the helix be taken in the convex surface of a cylinder standing on *H*, its horizontal projection is evidently a circle. In Fig. 56, let the circle $a^h p^h c^h$ be the horizontal projection of the curve, and *M* its axis. Let it be assumed that the generating point rises a distance $a^v b^v$ in making one complete revolution. Divide the circle $a^h p^h c^h$ and the line $a^v b^v$ into the same number of equal parts, say eight. Taking the generating point in the initial position at *a*, when the point has moved one-eighth the distance around the axis, it will be horizontally projected at p^h. But in this time it will have risen the height $a^v m$, or one-eighth of $a^v b^v$. Therefore the intersection of a line through *m*, parallel to *GL* with a line through p^h perpendicular to *GL*, gives the vertical projection p^v. Similarly, the projections of any number of points in the curve may be found.

If the convex surface of the cylinder containing the helix be developed on a plane surface, the helix will develop into a straight line.

The sinusoid shown in Fig. 56 is the vertical projection of the helix that crosses the elements of a cylinder of revolution at an angle of 45°.

In Fig. 56, the height $a_v b_v$ is called the pitch. The helix is used in designing screws, screw-propellers, winding-stairs, skew-arches, etc.

The angle which any element of the curve makes with *H*, is the angle at the base of a right triangle having the horizontal projection of the element for its base, and the difference of the heights of the extremities of the element for its altitude. But the angle made by this element with *H* is the angle which the tangent to the curve at that point makes with *H*, and evidently any arc of the curve is to its horizontal projection as an element is to its projection on *H*. Hence, to draw a tangent to the curve at any point *p*, a tangent is drawn to the circle at p^h, which will be the horizontal projection of the required tangent. The arc *pa* is then developed on $p^h r^h$, giving r^h, which is the horizontal trace of the required tangent. The vertical projection is $r^v p^v$.

In Fig. 57 is shown an approximate construction of a tangent to a plane curve, which is useful when no exact method is known. Let MN be an irregular curve, and t the point of tangency. Two or three points near to t, and on each side of it, are taken, and through these points and t secants are drawn as tn, to, tr, ts, etc. Then with t as center, any circle as xy cutting these secants is drawn. Along each of the secants, as pp', a distance from the circle $ap' = pt$ is laid off, giving the point p'. Connecting points found in this way gives the curve $p't's'$, which cuts the circle in t'. tt' is the required tangent.

A tangent is drawn to an irregular curve from an exterior point by passing a straight edge through the point so as to touch the curve. If the curvature of the curve is slight, the exact point of contact is not apparent. In Fig. 58 is shown a method for finding the point of tangency in such a case. Tt is a tangent to the curve MN. A secant, as ab, near Tt and parallel to it, is drawn. At a, a perpendicular to Tt is drawn downwards, and at b another is drawn upwards. The points c and d are found by taking $lc = bd = ab$. Similarly, other points as e, f, g, and h are found, and a line is drawn connecting these points. This line will intersect the curve in p, the required point of tangency.

PROB. 109. To find the projections of a circle lying in a plane whose traces are given.

PROB. 110. To find in Fig. 59 where the irregular curve ab intersects the planes of projection and the plane R.

PROB. 111. To construct the projections of a helix whose axis is inclined to both planes of projection.

CHAPTER IV.

SURFACES.

Art. 17. Generation and Classification of Surfaces.

33. A surface may be generated by a line moving according to some law. This line is called the *generatrix*, and the different positions of the line are *elements* of the surface. The kind of the surface is determined by the form of the generatrix and the law of its motion.

Surfaces may be grouped under two general classes. First: Surfaces that may be generated by right lines. These are called *ruled surfaces.* Second: Surfaces that can only be generated by curves. These have no rectilinear elements, and are called *double curved surfaces.*

There are three kinds of ruled surfaces. First: *Planes,* which may be generated by a right line intersecting and moving along another right line, always remaining parallel to its first position. Second: *Single curved surfaces,* which may be generated by a right line moving so that any two of its consecutive positions shall be in the same plane. Third: *Warped surfaces,* which may be generated by a right line moving so that no two of its consecutive positions shall be in the same plane.

There are three kinds of single curved surfaces. First: *Cylindrical* surfaces, in which all the positions of the rectilinear generatrix are parallel. Second: *Conical* surfaces, in which all the positions of the rectilinear generatrix meet in a point. Third: *Developable* surfaces, in which the consecutive positions of the rectilinear generatrix intersect two and two, no three positions intersecting in a common point.

Art. 18. Single Curved Surfaces and Tangent Planes.

34. Any *cylinder* may be generated by a right line moving so as to touch a curve and have all its positions parallel. The curve is the *directrix*, the moving line is the *generatrix*, and any position of the moving line is an *element* of the surface. The intersection of the cylinder by a horizontal plane is usually taken as the *base*. Cylinders are commonly named according to the forms of their bases. If the base is a circle, the cylinder is called a circular cylinder; if the base is an ellipse, the cylinder is said to be an elliptical cylinder, etc. If the base is a central curve, the line through the center of the base parallel to the rectilinear elements is called the *axis*. When the elements of a cylinder are perpendicular to the base, it is called a *right* cylinder, but if the elements are oblique to the base, it is an *oblique* cylinder.

35. The lines projecting a curve on one of the planes of the projection are elements of a right cylinder, and the base of this cylinder in the plane is a projection of the curve. The cylinder projecting the curve on *H* will be called the *H*-projecting cylinder, and the cylinder projecting the curve on *V*, the *V*-projecting cylinder. These two projections of a curve completely determine it.

36. A plane is tangent to a surface when it touches it in at least one point, if any plane passing through this point intersects the given surface and its tangent plane in lines that are tangent to each other. If it is required to find the plane tangent to a given surface at a given point of contact, any two plane curves on the surface which pass through the given point may be found, and the required tangent plane will be determined by the lines tangent to these curves at the given point.

If a plane contains two consecutive elements of a single curved surface, it will be tangent to the surface and tangent at every point along an element. Because any secant plane passed through any point of one of the consecutive elements will intersect the other in a consecutive point, and the line joining these consecutive points will lie in the tangent plane, and will be tangent to the curve cut from the surface.

If a curve and a tangent to it be projected on a plane, the projection of the right line will be tangent to the projection of the curve, because the projecting plane of the line will be tangent to the projecting cylinder of the curve, and the intersections of the plane of projection with the projecting plane and projecting cylinder will be tangent to each other.

PROB. 112. To construct the projections of a cylinder, and to assume a point on the surface.

In Fig. 60, $a^h b^h c^h$, the base in H, is first drawn. The center of the base, o, is the horizontal trace of the axis. The axis M is next assumed. S^h and T^h are drawn parallel to M^h and tangent to $a^h b^h c^h$. The base is projected on V in $d^v b^v$. Through d^v and b^v lines V^v and W^v are drawn parallel to M^v. The lines S, T, V, and W are called lines of *apparent contour.*

To find a point on the surface: Assume one projection of such a point as x^h. G^h, drawn through x^h parallel to M^h will be the horizontal projection of an element containing x. The intersection, k^h, of G^h with $a^h b^h c^h$ will be the foot of the element in H, which is vertically projected in k^v. G^v drawn through k^v parallel to M^v is the vertical projection of this element. x^v, the intersection of a perpendicular to GL through x^h with G^v, is the vertical projection of the point x. J^v is the vertical projection of another element which is horizontally projected in G^h, hence another point as z may be found which is horizontally projected in x^h.

PROB. 113. To pass a plane tangent to a cylinder at a point on the surface.

Analysis. If a plane be tangent to a cylinder at a point on the surface, it will be tangent to the cylinder all along the element. Any plane cutting this element will intersect the cylinder and the tangent plane in lines that will be tangent to each other. Then, if the base of the cylinder be taken in H, the line cut by H from the tangent plane must be tangent to the base which is cut from the cylinder; i.e., the horizontal trace of the tangent plane is tangent to the base of the cylinder. One point of the vertical trace of the tangent plane is found by drawing through any point of the element of contact a horizontal of the plane, and getting its vertical trace.

In Fig. 61, a is the given point on the surface, and M is the element of contact. HX, tangent to the horizontal projection of the base of the cylinder at k^h, the horizontal trace of M, is the horizontal trace of the tangent plane. A horizontal of the tangent plane through a intersects V in g^v, through which VX is drawn.

PROB. 114. To construct the projections of a right circular cylinder having its axis parallel to GL, to find a point on the surface, and to pass a plane tangent to the surface at this point.

PROB. 115. A cylinder whose right section is a circle of $\frac{3}{4}''$ radius has its axis inclined at 30° to H and 45° to V. To find the intersections of the cylinder with H, V and P.

PROB. 116. To pass a plane through a point without a cylinder tangent to the surface.

Analysis. Through the given point a line is drawn parallel to the elements of the cylinder. This line must lie in the tangent plane, and therefore its horizontal trace will be one point of the horizontal trace of the required plane. A line drawn through this point tangent to the base of the cylinder in H will be the required horizontal trace. The vertical trace may be found as in Prob. 113.

PROB. 117. To pass a plane tangent to a cylinder and parallel to a given right line.

Analysis. The required tangent plane must be parallel to the given line, and also parallel to the elements of the cylinder. Hence, if a plane be passed through the given line parallel to an element of the cylinder, it will be parallel to the required plane, and its traces will be parallel to the required traces. Therefore, a line tangent to the base of the cylinder, and parallel to the horizontal trace of this auxiliary plane, will be the required horizontal trace. The vertical trace may be obtained as before.

37. A *cone* may be generated by a right line moving so that it always passes through a given point and touches a given curve. The given point is the *vertex*, and the given curve is the *directrix*. The generatrix being indefinite in length, will generate two portions of the surface on opposite sides of the vertex, called the *upper* and *lower nappes*.

The intersection of the surface with H is usually taken as the *base*. Cones are named with reference to the forms of their bases. If the base is a circle it is called a circular cone; if an ellipse, it is called an elliptical cone, etc. If all the rectilinear elements make equal angles with a right line through the vertex, the cone is a *right cone* or a *cone of revolution*, and the right line is its *axis*. Evidently any plane through the vertex of the cone that intersects the surface will cut two elements from the cone.

PROB. 118. To draw the projections of a cone and to assume a point of the surface.

In Fig. 62, the base $a^hb^hd^h$ in H, and the projections of the vertex v, are assumed. Tangents to the base from v^h complete the horizontal projection of the surface. The base is projected vertically in a^vb^v, and lines drawn from v^v to a^v and b^v complete the vertical projection $a^vv^vb^v$ of the cone.

The elements S, T, V, and W are the elements of apparent contour.

A point on the surface is found by first assuming one of its projections as x^h. The line E^h drawn through v^h and x^h, will be the horizontal projection of two elements, either of which may contain the point x. The vertical projections of the elements are E^v and F^v. Therefore, the vertical projection of the required point may be x^v or z^v.

PROB. 119. To pass a plane tangent to a cone at a point on its surface.

Let the cone be given as in Fig. 63, with vertex o, and base bfc, and let p be the point of tangency. HX is drawn tangent to the base of the cone at f^h, the horizontal trace of the element of contact. One point, g^v, of the vertical trace is found as in Prob. 113.

PROB. 120. To pass a plane through a given point without a cone tangent to the surface.

Analysis. The tangent plane must contain an element of the cone, and hence must pass through the vertex. It also passes through the given point, and hence must contain the line connecting the given point with the vertex. The horizontal trace of this line will be a point in the horizontal trace of the required tangent plane, which is drawn tangent to the base of the cone. The vertical trace may be found as before.

PROB. 121. To pass a plane tangent to a cone and parallel to a given right line.

Analysis. If a line be drawn through the vertex of the cone parallel to the given line it will lie in the required plane, and hence its horizontal trace will be a point in the trace of the required tangent plane. Through this point a line tangent to the base will be the required horizontal trace. The vertical trace may be found as before.

PROB. 122. A right circular cone, the radius of the base being 1″ and altitude 2″, stands on *H*. If light comes from a point ½″ above the vertex and 1″ from the axis, to find the line of shade on the cone and the shadow on *H*.

PROB. 123. To construct the traces of a plane that shall make a given angle with one of the planes of projection and that shall contain a given line.

In Fig. 64, let M be the given line, and ϕ the given angle which the plane is to make with H. Assume any point as a of the given line, and draw through a the line A parallel to V and making the required angle with H. Revolve A about a^h—$a^v c^v$ as a vertical axis. The line A will describe a right circular cone whose base is $k^h l^h m^h$. Any plane tangent to this cone will make the required angle with H. The horizontal trace of M is t^h, and a line HX through t^h tangent to $k^h l^h m^h$ will be the required horizontal trace. The vertical trace is VX.

38. *Developable surfaces,* or the third kind of single curved surfaces, may be generated by moving a right line along a curve of double curvature so that in every position it is tangent to the curve. If w, x, y and z be consecutive points along a curve of double curvature, the tangent which contains the points w and x will intersect the consecutive tangent which contains the points x and y in x. The next consecutive tangent which passes through y and z will intersect the one just before it in y, but will not intersect the one through w and x. Thus each element of the surface intersects the preceding consecutive one, but no other preceding one lies in the same plane. Developable surfaces are so called because they will admit of development on a plane.

The principal surface of this kind is the *developable helicoid.* It may be generated by a right line moving so that it is always tangent to a helix.

PROB. 124. To construct elements of a developable helicoid and to assume a point on the surface.

In Fig. 65, let $abcd$ be the helical directrix. Since the rectilinear elements of the surface are successive tangents to the helix, as many elements as required may be constructed by Prob. 108.

The lower base, or the intersection of the surface with H, is evidently the involute of a circle. If a horizontal plane be passed at a height above H equal to the pitch of the helix, the intersection of this plane with the surface may be called the upper base.

To assume a point on the surface, a point may be taken on any element. Or, the horizontal projection of any point may be assumed as e^h, and a line S^h through e^h tangent to $a^hb^hc^hd^h$ will be the horizontal projection of the element containing e. The vertical projection of the element, S^v, may now be found, giving the required vertical projection at e^v.

PROB. 125. To construct the traces of a plane tangent to a developable helicoid at any point on the surface.

In Fig. 65, let e be the given point of tangency. The tangent plane is tangent to the surface all along the element S through e. Therefore, a tangent HX to the base at the horizontal trace of the line S will be the required horizontal trace.

The vertical trace VX may be obtained in the same manner as with the curved surfaces already given.

Prob. 126. To pass a plane through a point without a developable helicoid and tangent to the surface.

Analysis. With the given point as a vertex a cone of revolution is constructed whose elements make the same angle with H as the elements of the given surface. A plane tangent to this cone and to the helicoid will be the required plane. A line tangent to the bases of both surfaces must be the required horizontal trace. The horizontal trace determines parallel elements of contact in the two surfaces by which the required vertical trace can be readily found.

Prob. 127. To pass a plane tangent to a developable helicoid and parallel to a given right line.

Analysis. A plane is passed through the given line making the same angle with H that the elements of the given surface make. The horizontal trace of the required tangent plane will be parallel to the horizontal trace of the plane just found. The required horizontal trace will also be tangent to the base of the developable helicoid. The vertical trace may now be found as before.

Art. 19. Surfaces of Revolution.

39. A surface, which may be generated by a right line or a plane curve revolving about a right line in the same plane as an axis, is a *surface of revolution.*

The right circular cylinder and the right circular cone are the only single curved surfaces that are surfaces of revolution.

Evidently, the intersection of a surface of revolution by a plane perpendicular to its axis is a circle.

If a surface of revolution be intersected by a plane through the axis, the line of intersection is called a *meridian line.* The meridian line parallel to V is sometimes called the *principal meridian.*

A *normal line* to a surface at any point is a right line through that point perpendicular to the plane which is tangent to the surface at that point.

A *normal plane* to a surface at any point is a plane perpendicular to the tangent plane at that point.

Some of the principal double curved surfaces of revolution are the sphere, ellipsoids, paraboloid, hyperboloids and the annular torus.

40. Prob. 128. To draw the projections of a sphere, to assume a point on the surface, and to construct a plane tangent to the surface at any point of the surface.

Each projection of the sphere will evidently be equal to a great circle of the sphere, and its center will be one projection of the center of the sphere. The projections of a sphere may then be taken as in Fig. 66, $a^hb^hc^h$ being the horizontal, and $a^vd^vc^v$ the vertical projection.

To assume a point on the surface, one projection as e^h is first assumed. *HD* passing through e^h and o^h is the horizontal trace of the meridian plane through e. This plane is revolved about the vertical axis of the surface until it becomes parallel to *V*. After revolution, the meridian curve cut by *D* coincides with the principal meridian $a^hc^h — a^vd^vc^v$, and the revolved position of e is e_1. In the counter-revolution e_1 describes a horizontal arc which is vertically projected in $e_1{}^ve^v$. The vertical projection e^v of the required point is where a line through e^h perpendicular to *GL* intersects $e_1{}^ve^v$. f^v, similarly found, is the vertical projection of another point having e^h for its horizontal projection.

A plane *X* tangent to the surface at a point e will pass through the point e and be perpendicular to the radius of the sphere terminating in e.

Prob. 129. Given a sphere whose diameter is 3″ and whose center is in *H*, 1″ from *V*. To pass a plane tangent to the sphere and inclined 30° to *H* and 45° to *V*, and to find the point of tangency.

41. The *ellipsoid* is generated by revolving an ellipse about either axis. If the ellipse is revolved about its transverse axis, the surface is a *prolate* ellipsoid; if revolved about its conjugate axis, the surface generated is an *oblate* ellipsoid. The ellipsoids are sometimes called *spheroids*.

PROB. 130. To construct the projections of an ellipsoid, to assume a point on the surface, and to pass a plane tangent to the surface at a point on the surface.

In Fig. 67, M is the transverse axis which is taken as the axis of revolution perpendicular to H. The horizontal projection of the surface is the horizontal projection of the greatest *parallel*, *abc*, or circular section perpendicular to the axis. The vertical projection of the surface is the vertical projection of the principal meridian.

A point e of the surface is assumed as in PROB. 128.

To construct the tangent plane: The plane tangent to the surface at any point as e contains the tangent to the meridian curve at e and is also perpendicular to the normal to the surface at this point. G_1 is drawn tangent to the principal meridian at e_1, and H_1 is the normal at e_1. After the counter-revolution, the tangent takes the position G, and the normal comes to H. The normal intersects the axis in the point k. The tangent plane contains the tangent G, and is perpendicular to the normal H. Therefore, the required horizontal trace HX passes through t^h and is perpendicular to G^h, and the required vertical trace is perpendicular to H^v.

42. The *paraboloid* may be generated by revolving the parabola about its axis.

PROB. 131. To draw the projections of a paraboloid, to assume a point on the surface, and to construct a plane tangent to the surface at a point on the surface.

The best position for the axis is a position perpendicular to H or V.

A point is assumed on the surface, and a plane tangent to the surface at this point is constructed as in the preceding problem.

43. The *hyperboloid of revolution of two nappes* may be generated by revolving a hyperbola about its transverse axis.

PROB. 132. To draw the projections of a hyperboloid of revolution of two nappes, to assume a point on the surface, and to construct a plane tangent to the hyperboloid at a point on the surface.

A point is assumed on the surface, and a plane tangent to the surface at this point is constructed as in Prob. 130.

44. An *annular torus* may be generated by a circle revolving about an axis without the circle, and in the same plane with it.

Prob. 133. To draw the projections of an annular torus, to assume a point on the surface, and to construct a plane tangent to the torus at a point on the surface.

A point is assumed on the surface, and a tangent plane is constructed as in Prob. 130.

Prob. 134. To pass the surface of a sphere through any four points not in the same plane.

Analysis. If the four points be connected by three lines, and a plane be passed perpendicular to each of these lines at its middle point, three planes will be found which will pass through the center of the sphere. The right line connecting the center with one of the given points will be the radius whose true length is then obtained.

Prob. 135. To pass a plane through a given right line and tangent to a sphere.

Analysis. A secant plane is passed through the center of the sphere perpendicular to the given line, and its intersections with the line and sphere are found. From the intersection of the secant plane and the given line, two lines are drawn tangent to the great circle in which the secant plane cuts the sphere. A plane containing the given line and either one of these tangents to the spherical section will be a required tangent plane.

In Fig. 68, M is the given line, and c is the center of the given sphere.

1st. To pass a plane through c perpendicular to M. This plane is determined by the two lines G and I through c, constructed parallel respectively to the horizontal and vertical traces of the plane. The intersection h of the plane (G, I) with the line M is found by Prob. 45.

2d. To construct a tangent line from h to the circle cut from the sphere by the plane (G, I). If the plane (G, I) be revolved about G until it becomes parallel to H, the circle cut from the sphere by it will be horizontally projected in $k_1n^ho^h$, and the point h will revolve to h_1. The line J_1 from h_1, tangent to $k_1n^ho^h$ at k_1, will be the revolved position of a tangent line. After the counter-revolution of the tangent the point h_1 is at h, and m is in the axis of revolution; therefore $h^h m^h$ is the horizontal projection of the tangent, and k^h is the horizontal projection of the point of tangency. The point m is vertically projected in m^v, and $h^v m^v$ is the vertical projection of the tangent line.

3d. The plane of the lines M and J, or the plane X, is the required plane. The tangent plane is perpendicular to the radius that terminates at the point of tangency. Therefore HX must be perpendicular to L^h, and VX perpendicular to L^v.

Art. 20. Intersections and Developments.

45. The intersection of a right line or a curve of single curvature with a surface may be found by passing any plane through the line, intersecting the surface. The intersection of the given line with the line cut from the surface by the plane will be the required point. The plane should be selected so as to cut the surface in the simplest line possible.

The intersection of a plane with a surface may be found by cutting the given plane and surface by a system of auxiliary planes. Each auxiliary plane will cut from the given plane and surface lines whose intersections will be points of the required line.

The intersection of any two curved surfaces may be found by intersecting them by a system of auxiliary surfaces. Each auxiliary surface will cut from the given surfaces lines whose intersections will be points of the required line. The system of auxiliary surfaces should be assumed so that the simplest lines possible will be cut from the given surfaces.

46. A right line tangent to the curve of intersection of two surfaces at any point of the curve may be found by passing a plane tangent to each surface at this point. The intersection of these two planes will be the required tangent.

47. If a single curved surface be rolled upon a plane tangent to it, until all its rectilinear elements come successively into contact with the plane, the equivalent plane surface covered by the given surface will be the *development of the surface.*

Prob. 136. To find the intersection of a right circular cylinder by an oblique plane, to draw a tangent to this curve of intersection, to show the curve in its true size, and to develop the cylinder and the curve.

Analysis. Let the cylinder be given as in Fig. 69, *R* being the cutting plane. The cylinder and plane *R* are cut by a system of planes parallel to the axis of the cylinder and the horizontal trace of the cutting plane. Each of these auxiliary planes will cut from the cylinder two elements, and from the plane *R* a line parallel to *H*. The intersections of the elements cut from the cylinder and the line cut from *R* will be points of the required curve.

1st. The auxiliary plane *L* cuts the cylinder in elements projected vertically in I^v and J^v. This plane *L* cuts the plane *R* in the line vertically projected in K^v. The intersections of line *K* with lines *I* or *J*, or *m* and *n*, are two points of the required intersection. Similarly, any number of points may be found.

It may be shown that a plane through the axis of the cylinder perpendicular to the plane *R*, intersects *R* in a line which contains the transverse axis of the elliptical section.

2d. To draw a tangent to the curve at any point *n*. A plane is passed tangent to the cylinder at this point. The horizontal trace of this tangent plane is *HT*. The intersection of this tangent plane with the plane *R*, is *T*, the tangent line required.

3d. To find the curve of intersection in its true dimensions, the plane *R* is revolved about *HR* into *H*, and the revolved positions of points on the curve are found. T_1 is the revolved position of the tangent *T*.

4th. It is assumed that the cylinder is cut along the element *J*, opposite to the element through *g*. In Fig. 69*a*, the element through *g* is taken at kg_1. The cylinder develops into a rectangle *ablo*, whose altitude *ao* is equal to the altitude of the cylinder, and whose base *lo* is equal to the rectified base of the cylinder. The developed position of any point *n* of the curve of intersection is found by laying off from *k* to n_1 the rectified arc $g^h n^h$, and erecting a perpendicular to *lo* at n_1. The developed position n' is at a distance from *lo*, equal to the distance from n^v to *GL*.

Prob. 137. To find the intersection of a plane with an oblique cylinder, to draw a tangent to the curve of intersection at any point, and to show the curve in its true dimensions.

Analysis. The plane and cylinder are intersected by a system of planes parallel to the elements of the cylinder and perpendicular to *H*. Each of these planes will intersect the cylinder in two elements, and the given plane in a line, whose intersections are points of the required curve.

A tangent to the curve at any point is the intersection of a plane tangent to the cylinder at that point and the given plane.

The curve in its true dimensions may be found as in the preceding problem.

Prob. 138. To find the intersection of a plane with a right circular cone, to draw a tangent to the curve of intersection at any point, and to develop the cone and the curve.

Let the cone be given as in Fig. 70, and let *R* be the given plane.

1st. Planes are passed through the vertex of the cone perpendicular to *V*. Each of these planes will cut the cone in two elements, and the given plane in a line whose intersections are two points of the required curve. Let one of the planes be *D*. It cuts from the cone elements horizontally projected in I^h and J^h, and it cuts the plane *R* in the line horizontally projected in K^h. The intersections *e* and *f* of the line *K* with the lines *I* and *J* are two points of the required curve.

2d. To draw a tangent to the curve at any point, as *f*. A plane, whose horizontal trace is *HU*, is passed tangent to the cone at the point *f*. The intersection *T* of this plane with plane *R* is the required tangent.

3d. To develop the surface. It is assumed that the surface is cut along any element, as *ca*, and rolled on the plane tangent at this element. With any point, as c' in Fig. 70*a*, as a center, and with $c'a' = c^va^v$ as a radius, an indefinite arc $a'a''$ is drawn. The circle $a^hb^hd^h$ is developed from a' to a''; then the circular section $c'a'a''$ is the development of the cone.

To find the development of any point as *e*, in the curve of intersection, the element *ce* is revolved about the axis of the cone until it is parallel to *V*. Then it will coincide with the element *ca*, and the true length of *ce* is shown at c^ve_1. If the arc a^hg^h be developed from a' to g' on $a'a''$, then $c'g'$ will be the development of the element *cg*. And the distance c^ve_1, measured from c' along $c'g'$, will give e', the developed position of *e*.

Prob. 139. To pass a plane so as to cut a conic section from a right circular cone, and to show the curve in its true dimensions.

A plane parallel to the base will intersect the cone in a *circle*. If the cutting plane makes a smaller angle with the base than the elements do, the curve is an *ellipse*. If the secant plane is parallel to an element, it cuts the cone in a *parabola*; and if the plane makes with the base a greater angle than the elements, the curve is a *hyperbola*.

Prob. 140. To find the intersection of a plane with any oblique cone, and to draw a tangent to the curve of intersection at any point of the curve.

Analysis. Auxiliary planes intersecting the cone are passed through the vertex of the cone perpendicular to *H*. Each of these planes cuts two elements from the cone, and a line from the given plane whose intersections are points of the curve.

A tangent to the curve of intersection at any point may be constructed as in Prob. 138.

Prob. 141. To find the intersection of a plane with any surface of revolution, and to draw a tangent to the curve of intersection.

Analysis. Auxiliary planes are passed perpendicular to the axis. Each of these planes cuts a circle from the given surface, and a right line from the given plane, whose intersections are points of the required curve. If the axis be taken perpendicular to one of the planes of projection, the projections of the circular intersections are readily found.

A tangent to the curve of intersection is found as before by passing a plane tangent to the surface at the given point, and finding the intersection of this tangent plane with the given plane.

Let the surface be the annular torus given as in Fig. 71, and let R be the secant plane. The auxiliary horizontal plane whose vertical trace is VL, intersects the surface in two circles, horizontally projected in G^h and J^h, and intersects the plane R in a line horizontally projected in K^h. The intersections e, f, g, and h of K, with G and J, are points of the required curve. Similarly, any number of points may be found.

To draw a tangent to the curve of intersection at any point g. By Prob. 130, the tangent plane to the surface at g is X. The tangent plane intersects the plane R in the required tangent T.

PROB. 142. To find the intersection of a triangular pyramid with a sphere.

PROB. 143. To find the intersection of a cone and a cylinder, and to draw a tangent to the curve of intersection at any point.

Analysis. Auxiliary planes cutting both surfaces are passed through the vertex of the cone, and parallel to the rectilinear elements of the cylinder. Each of these planes will intersect each of the surfaces in two elements whose intersections are points of the required curve.

Let the surfaces be given as in Fig. 72. A line G, through the vertex of the cone parallel to the elements of the cylinder, will be a line of each auxiliary plane. The horizontal trace n^h of this line will be a point of the horizontal trace of each secant plane. Through n^h, any line, as HL, is drawn intersecting both bases. HL is taken as the horizontal trace of a plane that intersects the cylinder in the elements I and J, and the cone in the elements K and D. These elements intersect in e, f, g and *h*, which are four points of the required curves. Similarly, any number of points may be found.

The tangent to a curve of intersection at any point, as *f*, may be found by passing a plane tangent to each surface at *f*, and getting the intersection of these two tangent planes. *HU* and *HK* are the horizontal traces of these planes, and the required tangent line *T* is the intersection of these planes.

The plane *E*, tangent to the cylinder along the element *A*, cuts from the cone the two elements *B* and *C*. The elements of the cone between *B* and *C* in front do not intersect the cylinder; hence the curves of intersection are tangent to the elements *B* and *C* at the points *l* and *f*. Similarly, the plane *F*, tangent to the cylinder on the other side, cuts two elements *M* and *H* from the cone that are tangent to the curves.

When two auxiliary planes, as *E* and *F*, tangent to the cylinder, both intersect the cone, it is evident that the cylinder penetrates the cone, giving two distinct curves of intersection. If one of these tangent planes intersects the cone, and the other does not, there will be a single curve of intersection. If neither of these tangent planes intersects the cone, and the cone lies between them, the cone penetrates the cylinder, giving again two distinct curves of intersection.

PROB. 144. Let the base of the cylinder be in *H* and the base of the cone in *V*. To find the intersection and tangent as in the preceding problem.

PROB. 145. To find the intersection of two cylinders, and to construct a tangent to the curve of intersection.

Analysis. Auxiliary planes may be passed parallel to the elements of both cylinders. Each auxiliary plane that intersects the cylinders will cut two elements from each, whose intersections will be four points of the required intersection.

Let the cylinders be given as in Fig. 73. Through *a*, one point of the element *G*, a line *A* is drawn parallel to the elements of the other cylinder. *HS* connecting n^h and m^h, the traces of *G* and *A* respectively, will be the horizontal trace of a plane parallel to the elements of both cylinders. Any line as *HU* parallel to HS, and intersecting the bases of both cylinders, may be taken as the horizontal trace of an auxiliary plane. The plane U cuts the cylinders in the elements I, J, K and H, whose intersections c, f, g, and h are four points on the required curves. Similarly, any number of points may be found.

The tangent T to the curve at f may be obtained as in the preceding problem.

PROB. 146. To find the intersection of two cones, and to draw a tangent to the curve of intersection at any point.

Analysis. Auxiliary planes may be passed through the vertices of both cones. The horizontal trace of the line connecting the two vertices will be a point common to the horizontal traces of all the auxiliary planes. Any line through this point intersecting the bases of both cones may be taken as the horizontal trace of a plane cutting two elements from each cone, whose intersections are points of the required curve.

A tangent to the curve at any point may be constructed as in the preceding problem.

PROB. 147. To find the intersection of two surfaces of revolution whose axes are in the same plane, and to draw a tangent to the curve of intersection at any point.

Analysis. If the axes of the two surfaces are parallel, an auxiliary plane perpendicular to the axes will cut circles from the surfaces whose intersections will be points of the required curve. But if the axes intersect, a system of auxiliary spheres having their centers in the point of intersection of the axes will cut both surfaces in circles perpendicular to the axes. The points of intersection of the two circles cut by such a sphere will be points of the required curve of intersection.

Let the surfaces be the ellipsoid and paraboloid whose projections are taken as in Fig. 74. The axis of the ellipsoid is perpendicular to H, and the axis of the paraboloid is parallel to V, and the axes intersect in a. With a^v as a center, and a^vb^v as a radius, the arc $b^vc^vd^v$ is described. The arc $b^vc^vd^v$ may be taken as the vertical projection of an auxiliary sphere. This sphere cuts from the ellipsoid the circle $b^hc^hc^h - b^vc^v$, and from the paraboloid the circle vertically projected in d^vk^v. These circles intersect in two points, e and f^h-e^v, which are points of the required curve. Similarly, any number of points may be found.

A tangent may be drawn to the curve of intersection at any point e, as in the last three problems, but another method will be given. If a normal line to each surface be drawn at the assumed point e, the plane of these normals will be perpendicular to the two planes, each tangent to one of the surfaces at e, and hence perpendicular to their intersection, which is the required tangent line.

A normal to the ellipsoid at e is constructed by revolving the meridian plane containing e about the axis of the surface until it becomes parallel to V. The meridian plane will thus be vertically projected in $b^vl^vc^v$, and e in b^v. A perpendicular G' to the tangent to $b^vl^vc^v$ at b^v is the revolved position of the normal. After the counter-revolution, the projections of this normal are $a^hc^h-j^ve^v$. Similarly, the normal K to the paraboloid at the point e may be constructed. The line j^vm^v is the vertical projection of the intersection of the plane of these normals with the plane of the axes, and is hence parallel to the vertical trace of the plane of the normals. Therefore, e^vt^v, perpendicular to j^vm^v, is the vertical projection of the required tangent. The line e^hn^h is the horizontal projection of the intersection of the plane of the normals with the horizontal plane whose vertical trace is VL, and is hence parallel to the horizontal trace of the normal plane. Therefore, e^ht^h, perpendicular to e^hn^h, is the horizontal projection of the required tangent.

PROB. 148. Given two spheres, diameters 2″ and 3″, with centers 2″ apart on a line making angles of 30° and 45° with H and V. To find the traces of the plane and the radius of the circle in which the spheres intersect.

PROB. 149. To find the intersection of a cylinder and sphere.

Analysis. When the surfaces are given, as in Fig. 75, they may be intersected by auxiliary planes parallel to the elements of the cylinder and perpendicular to *H*. Each plane will cut a circle from the sphere and two elements from the cylinder, whose intersections will be points of the required curve. If each secant plane be revolved about its horizontal trace into *H*, the circle cut from the sphere will be shown in its true dimensions.

A tangent to the curve of intersection may be found as before.

PROB. 150. To find the intersection of a cone and hemisphere, and to draw a tangent to the curve of intersection at any point.

Analysis. Auxiliary planes may be passed through the vertex of the cone perpendicular to *H*. Each of these planes will cut from the cone two elements, and an arc of a circle from the hemisphere, whose intersections will be points of the required curve.

A tangent to the curve may be found as before.

PROB. 151. In Fig. 76 are shown the projections of a hexagonal washer. *M* is the axis of a cylindrical hole of $\frac{1}{4}''$ radius, which is bored through the washer. To find the intersections.

PROB. 152. A "*V*" threaded screw is generated by revolving an equilateral triangle about an axis in the same plane, one side of the triangle being parallel to the axis. The side of the triangle is $\frac{1}{2}''$, and the screw has $2''$ pitch and $3''$ outside diameter. To find the intersection of a plane parallel to the axis, and $1''$ from it.

Art. 21. Warped Surfaces.

48. In warped surfaces the movement of the generatrix is controlled by three conditions.

The *directrices* are lines or surfaces which the generatrix or moving line must touch. The *director* is a surface to which the generatrix must remain parallel.

In the common warped surfaces the directrices are lines, and the director is a plane.

Warped surfaces may be classified as follows: —

I. Warped surfaces having a plane director and two directrices.

II. Warped surfaces having three directrices.

III. Warped surfaces having a cone director and two directrices.

49. The simplest warped surface is one of the first class, the *hyperbolic paraboloid.* It may be generated by a right line moving so as to touch two rectilinear directrices not in the same plane, and always remaining parallel to a given plane director. This surface is called the hyperbolic paraboloid, because intersecting planes cut hyperbolas and parabolas from it. As the directrices approach parallelism, the surface approaches a plane as a limiting surface.

This surface is frequently employed. It is often used in masonry construction. The cow-catcher of a locomotive is usually in the form of this surface.

Prob. 153. Given the plane director and two rectilinear directrices of a hyperbolic paraboloid, to construct elements of the surface, and to assume any point on the surface.

Analysis. An element of the surface may be constructed by passing a plane through any assumed point of one directrix parallel to the plane director. The line connecting the assumed point with the point in which this plane intersects the other directrix will be an element of the surface.

A point of any warped surface may be assumed by taking a point on an element; or, the horizontal projection of a point may be assumed and a perpendicular to H erected at this point. Then a plane is passed through this perpendicular, cutting the surface in a line. The intersection of this line with the perpendicular is the required vertical projection.

In Fig. 77, let lines M and N be the directrices, and R the plane director. Construct any two lines in the plane director as I_1 and J_1. Then through any point a of N, draw lines I and J respectively parallel to I_1 and J_1. The lines I and J determine a plane parallel to R, which intersects the H-projecting plane of M in a line kl. The line kl intersects M in the point e, which is the point of intersection of the plane (I, J) with M. The line S, connecting a and e, is the required element. Similarly, any number of elements may be constructed.

Prob. 154. To construct elements of the hyperbolic paraboloid by proportional division of the directrices, and to draw an element parallel to a given line.

Analysis. A system of parallel planes will divide a right line into parts which are proportional to the corresponding divisions made by these planes with another right line. Hence, if the corresponding points of proportional division on the two given lines be connected by right lines, each of these lines will intersect both directrices, and be parallel to a plane which may be taken as the plane director.

To construct an element parallel to a given line, this line being parallel to the plane director. Through one of the directrices a plane is passed parallel to the given line, and the intersection of this plane with the other directrix is found. A line through this point of intersection, parallel to the given line, will be the required element.

In Fig. 78, let *M* and *N* be the given directrices. The line *M* is divided into equal parts, by beginning at any point as *b*, and laying off successively distances equal to *bc*. Likewise, the line *N* is divided, by beginning at some point as *a*, and laying off successively equal distances of any length as *ad*. Then lines *ab*, *cd*, etc. will be elements of a hyperbolic paraboloid.

The hyperbolic paraboloid is one of two *doubly ruled* surfaces. A doubly ruled surface is one in which there may be two different sets of rectilinear elements. Either directrix of one set of elements may become the generatrix of a second set. Thus each element of one generation intersects no element of its own set, but meets all the elements of the other set. Elements of the second generation may be constructed by taking any two elements of the first generation as directrices, and a plane director parallel to the directrices of the first generation.

If any two rectilinear elements of the same generation be taken as directrices with a plane director parallel to the first directrices, and a surface be generated, it will be identical with the first surface. In order to prove this, it is sufficient to demonstrate that any point of an element of the second generation is also a point of the first.

In Fig. 79, let M and N be directrices and E and F elements of the first generation, and let the plane director be H. With E and F as the new directrices, and a plane director parallel to M and N, an element S of the second generation is constructed. Through j any point of S, a horizontal plane is passed whose vertical trace is VL. The plane L will cut the directrices M and N in k and l, giving the line kl, an element of the first generation. Next, lines G and I are drawn through a and k parallel to S intersecting H in m and n. Likewise, lines D and C are drawn through b and l parallel to S. Line mt is drawn intersecting S in h. m, n and o are in the same right line as are also r, s and t. The plane (M, G) will be parallel to the plane (N, D), because both planes are parallel to the plane director of the second generation, hence their horizontal traces m^ho^h and r^ht^h will be parallel. The line n^hs^h is then drawn. As G and I are parallel,

$$\begin{aligned} & ok : ka :: o^hn^h : n^hm^h. \\ \text{also} \quad & rl : lb :: r^hs^h : s^ht^h ; \\ \text{but} \quad & ok : ka :: rl : lb ; \\ \text{therefore} \quad & o^hn^h : n^hm^h :: r^hs^h : s^ht^h, \end{aligned}$$

and consequently n^hs^h must pass through h^h. The plane containing the three parallel lines I, S and C contains also the element kl which therefore must intersect the element S at j. Thus it is proved that through any point j of the hyperbolic paraboloid, two rectilinear elements, one of each generation, may be passed.

PROB. 155. To pass a plane tangent to a hyperbolic paraboloid at a given point on the surface.

Analysis. Through the given point pass an element of each generation. A plane tangent to the surface at the given point will contain the line tangent to the curve which any plane through the point cuts from the surface. If a plane be passed through the point cutting a rectilinear element from the surface, the line of intersection is this element, and the line tangent to the line of intersection must therefore coincide with this element. Hence, the tangent plane is determined by the two elements passing through the point.

In Fig. 80, let M and N be the directrices and H and O any two elements, and assume any point of the surface as a. Through a draw ah and aj parallel respectively to lines H and O. These lines determine a plane through a parallel to the plane director. This plane intersects N in the point c; therefore, ac is an element of the first generation through a. Now O and H are taken as directrices, and M and N as elements for the second generation, and through a lines I and J are drawn parallel respectively to N and M. These lines determine a plane through a parallel to the plane director of the second generation. This plane intersects H in the point f; hence af is an element of the second generation through a. X, the plane of the lines ac and af, is the required tangent plane.

50. The *hyperboloid of one sheet* is a warped surface which may be generated by a right line revolving about an axis not in the same plane with it. This is a doubly ruled surface, and is the only warped surface of revolution.

PROB. 156. To construct elements of the hyperboloid of revolution of one nappe by revolving a right line about another not in the same plane, and to construct elements of both generations.

In Fig. 81, let M be the axis, and assume the line N parallel to V as one position of the generatrix. As the generatrix revolves about the axis, each point of it describes a horizontal circle. The point a of the generatrix, which is the nearest point to the axis, describes the circle man, which is called the *circle of the gorge*. $M^h a^h$ is the radius of the gorge circle, and shows the distance from the axis to the generatrix. As the generatrix in every position is at this constant distance from the axis, the horizontal projection of the generatrix in any position must be tangent to the circle $m^h a^h n^h$. The horizontal trace k of the generatrix N describes a circle $k^h l^h c^h$ on H which is the intersection of the required surface with H.

An element may be assumed by drawing a tangent E^h to the circle $m^h a^h n^h$ at any point as h^h. E^h may be taken as the horizontal projection of an element whose horizontal trace j^h is in the circle $i^h k^h j^h$. The vertical projection of h is at h^v in the vertical projection of the gorge circle, and j^v is in GL. Hence $h^v j^v$ is the vertical projection of the element E.

If through the point a, another line O be drawn parallel to V and making with H an angle equal to the angle made by N, it will generate when revolved about the axis M, the same hyperboloid. A horizontal plane whose vertical trace is VL, will cut N and O in the points d and e. As the generatrices N and O revolve, the points d and e will generate the same circle. Likewise, the two points in which any horizontal plane intersects N and O will generate the same circle ; therefore, the two generations will give the same surface. As the surface may be generated by two different right lines, it follows that through any point of the surface, two elements may be drawn.

If the first generatrix remains fixed in its initial position N, and the second generatrix O be revolved, the point e will at some time in its motion coincide with d, and O in this position will be an element of the second generation intersecting the first generatrix in d. In the same way O can be revolved until any other point in it coincides with a point in N, giving another element of the second generation intersecting N. But any element of the first generation may be taken as the first generatrix ; therefore, each element of one generation will intersect all the elements of the other generation.

But three rectilinear directrices, no two of which are parallel, are sufficient to determine the motion of a generatrix. Hence, if any three elements of one generation be taken as directrices, and an element of the other generation be moved so as always to touch them, the same surface will be generated.

Any warped surface with three rectilinear directrices, no two of which are parallel, is a *hyperboloid of one nappe*. The *hyperboloid of revolution of one nappe*, the one constructed in Fig. 81, may be defined as the warped surface generated by a right line moving so as to touch three other right lines equally distant from a fourth (the axis), the distances being measured in a plane perpendicular to the fourth, the directrices always making equal angles with this plane.

This hyperboloid of revolution is the only one employed in the arts.

PROB. 157. To construct a plane tangent to a hyperboloid of revolution at any point of the surface, and to find a meridian curve of the surface.

In Fig. 81, assume t^h as the horizontal projection of a point on the surface. A^h and B^h drawn through t^h tangent to $m^h a^h n^h$, are the horizontal projections of two elements through t. The plane N passed through A and B is the required tangent plane.

A meridian curve of the surface is constructed by passing a plane through the axis parallel to V, and finding its intersection with the elements. The horizontal trace of this plane is HU, and an element E intersects it in the point r, which is one point in the meridian curve. If through r^h a line C^h be drawn tangent to $m^h a^h n^h$, it will be the horizontal trace of an element C which intersects the plane U in the point $(r^h - s^v)$. The points r^v and s^v are seen to be equally distant from $m^v n^v$. Thus it is shown that for every point as r^v on the curve on one side of $m^v n^v$, there is a point, as s^v on the opposite side, equally distant from $m^v n^v$, and therefore $m^v n^v$ is an axis of symmetry.

A horizontal plane whose vertical trace is VL intersects N in d, and $d^h g^h$ is the horizontal projection of the circle generated by this point, and g is the point of intersection of the meridian plane and this circle. The distance $d^v g^v = M^h d^h - a^h d^h$ is the difference between the hypothenuse and altitude of a right triangle having $M^h a^h$ for its base. If a point of the curve is taken farther from $m^v n^v$ than g^v is, the base of the corresponding triangle remains of the same length, while both the hypothenuse and altitude increase. If b = base, y = altitude, and z = hypothenuse, then $z^2 - y^2 = b^2$, and $z - y = \frac{b^2}{z + y}$ . . . (1).

In (1), as b is constant, $z - y$ will diminish toward zero as a limit, as z and y increase indefinitely, and $z - y$ is the distance between the line N and the curve measured on a horizontal line. Hence, N is an asymptote to the curve. Likewise it might be shown that O is another asymptote.

The element E intersects N in p and O_1 in o. The vertical projection E^v must be tangent to the curve $r^v n^v s^v$ at r^v, for the element has but one point in common with the curve, and no point of the element can be vertically projected on the right of the curve. As $r^h o^h = r^h p^h$, then $r^v o^v = r^v p^v$, or the point of contact r^v bisects the tangent between the asymptotes. It is proved in Analytical Geometry that this is a property peculiar to the hyperbola. Therefore the meridian of a hyperboloid of revolution is a hyperbola. It is now evident that the surface may be generated by revolving a hyperbola about its conjugate axis. The surface receives its name from this method of generation.

51. The *helicoid* is a warped surface which may be generated by a right line moving so as to intersect a helix and its axis, and making always a constant angle with the axis.

PROB. 158. To construct elements of a helicoid and to pass a plane tangent to the surface at any point.

In Fig. 83, the axis M is taken perpendicular to H, and N is the generatrix in a position parallel to V. As the generatrix moves around the axis and also in the direction of its length, each point of the generatrix will generate a helix. The helix generated by the point a is ajk.

To construct an element, the horizontal projection E^h passing through M^h is first assumed. The point l is the intersection of the element with the helix. The distance $l^v x$, from l^v to GL, is laid off from b^v to y, along the vertical projection of the axis. Then the line $l^v y$ will be the required vertical projection E^v.

To pass a plane tangent to the surface at any point as l. The tangent plane must contain the element E, which passes through l, and also the tangent to the helix at that point. The tangent to the helix at l is T. The plane X, which contains the lines E and T, is the required tangent plane.

The elements intersect H in the curve $a^h m^h n^h$, which is the spiral of Archimedes.

When the elements of a helicoid are perpendicular to the axis, the surface is called a *right helicoid.* If the elements are oblique to the axis, the surface is called an *oblique helicoid.*

The curved surface of the threads of an ordinary V-screw is an oblique helicoid. The right helicoid is the working surface of the square-threaded screw, and of the common screw propeller.

Art. 22. Additional Problems on Surfaces.

52. Prob. 159. To construct elements of a warped surface with any three rectilinear directrices.

Analysis. Through any assumed point of the first directrix a plane may be passed through the second directrix. The intersection of this plane with the third directrix will be a point, which connected with the assumed point, gives a line which intersects all three directrices, and hence is an element of the surface.

Prob. 160. To construct elements of a warped surface with three linear directrices, one or more being curved lines.

Analysis. Through any assumed point of the first directrix and different points of the second directrix, lines may be passed. These lines form, in general, a conical surface having its vertex at the assumed point. The intersection of this cone with the third directrix will be a point, which connected with the assumed point, gives a line which intersects all three directrices, and hence is an element of the surface.

The construction of an element, when the directrices are all curvilinear, is seen in Fig. 84.

53. A *conoid* is a warped surface with a plane director and two linear directrices, one being a right line and the other a curve. A *right conoid* is one whose rectilinear directrix is perpendicular to the plane director. This directrix is called the *line of striction.*

Prob. 161. To construct a conoid and a plane tangent to the surface at any point.

Analysis. If a plane be passed through an assumed point of the rectilinear directrix parallel to the plane director, it will intersect the curved directrix in a point or points which connected with the assumed point will give one or more elements of the surface. The construction will be simplified by taking the curvilinear directrix in *H*.

A plane, tangent to the surface at any point, will contain the rectilinear element through this point. If the curvilinear directrix is in *H*, the horizontal trace of this tangent plane will be tangent to the curved directrix at the foot of the element of contact.

54. The *cylindroid* is a warped surface having a plane director and two curvilinear directrices.

This surface has been used for the soffit of a descending arch.

PROB. 162. To construct elements of a cylindroid.

55. The *warped arch* or *cow's horn* is a warped surface having one rectilinear and two curvilinear directrices; the curved directrices are equal circles in parallel planes, and the straight directrix is perpendicular to the planes of the circles, and passes through the middle point of the line joining their centers.

The soffit of the skew arch is frequently made to conform to this surface.

PROB. 163. To construct elements of a warped arch.

The elements may be constructed as in Prob. 160. The construction will be simplified by taking the circular directrices parallel to *V*.

56. The *elliptical hyperboloid* is a warped surface having as directrices three similar ellipses, one of these ellipses being midway between two larger equal ellipses, all having their corresponding axes parallel, and their centers in a right line perpendicular to their planes.

PROB. 164. To construct an elliptical hyperboloid, and to pass a plane tangent to the surface at any point.

The surface may be constructed as in Prob. 160. The construction will be simplified by taking one of the larger ellipses in *H*.

A symmetrical construction, such as is seen in Fig. 85, may be made as follows: An ellipse is drawn in *H* as in Prob. 97, by first taking equidistant points on one of the concentric circles. A line connecting any two of the corresponding points, found on the ellipse, may be assumed, as the horizontal projection of an element, and the horizontal projection of any other element will connect two points on the ellipse corresponding to two points on the circle at the same distance from each other as the first two. The height above *H* of the plane of the upper base may now be assumed, and the vertical trace of this horizontal plane drawn. The vertical projections of the elements may now be readily constructed.

This is a doubly-ruled surface, and therefore through any point of it two elements may be drawn. A tangent plane to the surface at any point is determined by the two elements that may be passed through this point.

57. The *Serpentine* is a double curved surface that may be generated by a circle moving about an axis, so that it is always in the same plane with the axis, its center describing a helix. It may also be generated by a sphere whose center moves in a helix. The serpentine may be regarded as the envelope of the generating sphere.

The serpentine and the torus are sometimes called tubular surfaces.

PROB. 165. To construct the projections of a serpentine, and to pass a plane tangent to the surface at any point.

Analysis. The projections of a helix are first drawn. Then a series of circles whose diameters are equal to the diameter of the generating sphere are drawn with their centers on the vertical projection of the helix, and a corresponding series with their centers on the horizontal projection of the helix. The lines enveloping the series of circles in each projection will be lines of visible contour.

To pass a plane tangent to the surface at any point. A horizontal plane through the assumed point, and a plane through the point and axis, will cut circles from the surface. The right lines tangent to these circles at the given point will determine the required tangent plane.

58. A *right* hyperbolic paraboloid is one in which the plane directors of the two generations are perpendicular. A *line of striction* in a warped surface is the shortest line connecting consecutive elements. In the hyperbolic paraboloid it is that element of one generation which is perpendicular to the plane director of the other generation.

PROB. 166. To construct a right hyperbolic paraboloid and a line of striction.

The construction will be simplified by taking H and V as the two plane directors, and a line in H as one of the directrices of one generation, and a line in V as one of the directrices of the other generation.

PROB. 167. To construct the projections of a developable helicoid, the projections of whose axis make given angles with GL.

PROB. 168. To inscribe a sphere in a given triangular pyramid.

PROB. 169. To pass a plane through the point a in Fig. 86, tangent to the spheres whose centers are b and c.

PROB. 170. To pass a plane so as to cut a parabola or a hyperbola from a hyperbolic paraboloid.

Any plane parallel to the line of intersection of the two plane directors will cut a parabola, and any plane oblique to this intersection will cut a hyperbola from the surface.

PROB. 171. To find the intersection of an oblique plane with an oblique cone, and to develop the cone and the curve of intersection.

Analysis. To develop the cone: If an auxiliary sphere be passed having its center at the vertex of the cone, it will cut the cone in a curve all of whose points will be at the same distance from the vertex. Hence, when the cone is developed this curve will develop into an arc of a circle whose radius is equal to the radius of the sphere. On this arc the curve of intersection may be developed, and the developed position of any element is found by connecting the developed position of its intersection with the sphere and the vertex. The developed position of any point in the required intersection is obtained by getting the development of the element containing the point, and measuring along it the true distance from the vertex to the point.

59. A plane passed through the axis of an oblique cylinder or cone having a central curve for its base, and perpendicular to the base, is called the *principal plane.*

Sub-contrary sections of such a cylinder or cone are cut by two planes perpendicular to the principal plane, and making equal angles with the plane of right section.

Prob. 172. To pass two planes cutting sub-contrary sections from an oblique cylinder or cone with a circular base.

Prob. 173. To find the intersection of a plane and a helicoid, and to draw a tangent to the curve of intersection.

Analysis. Auxiliary planes may be passed through the axis of the helicoid. Each auxiliary plane will cut a rectilinear element from the helicoid and a right line from the given plane, whose intersection will be one point of the required curve.

Prob. 174. Fig. 87 shows the projections of a bolt head, partly square and partly conical; and Fig. 88 shows a hexagonal bolt head with its end turned to a spherical shape. To find the curve e^vf^v in each.

Prob. 175. To find the intersection of a plane and any of the curved surfaces that have been given.

Prob. 176. To find when a line, whose projections are given, intersects any given curved surface.

Prob. 177. To find the intersection of an oblique circular cylinder, whose axis makes given angles with the planes of projection, with any of the curved surfaces that have been given.

Prob. 178. To find the intersection of a given oblique prism with any of the curved surfaces that have been given.

Prob. 179. To construct the two projections of a right groined vault, one foot in thickness, when the cylindrical axes are in the same plane parallel to *H*, and equally inclined to *V*.

Analysis. A right groined vault is formed by the intersection of two equal right circular semi-cylindrical shells. The axes of the semi-cylindrical shells are at right angles to each other. Each shell is the volume between two concentric semi-cylinders differing one foot in radius, and standing on a horizontal plane through its axis.

The groined vault or arch is much used in masonry constructions, especially in architecture.

PROB. 180. In Fig. 90 are shown the plan, front and end elevations of a round rod turned in an ordinary lathe from a rectangular bar. It is required to find h^h-g^h and e^v-f^v, the projections of the lines of visible contour of this surface of revolution connecting the round rod and rectangular bar.

PROB. 181. To pass a plane tangent to a helicoid and parallel to a given right line.

Analysis. Let the helicoid be given as in Fig. 83. A right circular cone with any point of the given line as a vertex, whose elements make the same angle with H as the elements of the helicoid, is constructed. Then a plane passed through this line tangent to the cone, will be parallel to the required plane.

PROB. 182. To pass a plane tangent to a helicoid, and perpendicular to a given line.

Analysis. A cone of revolution is constructed with any point of the axis as a vertex, whose elements make with H the same angle as the elements of the helicoid. Then a plane passed through the vertex of this cone perpendicular to the given line, will, if the construction be possible, cut from the cone two elements each of which is parallel to an element of the helicoid, and having the same horizontal projection. If a plane be passed through either of these elements parallel to the auxiliary plane, it will be tangent to the surface, and perpendicular to the given line.

PROB. 183. To pass a plane tangent to two oblique cones having a common vertex, the bases being in the planes of projection.

PROB. 184. To construct the projections of one coil of a screw conveyor of the form of a developable helicoid. The diameter of the inner helix being 6″, of the outer 12″, pitch 20″.

PROB. 185. To pass a plane through a point without the surface, and tangent to any one of the given double curved surfaces of revolution.

Analysis. Through a point without the surface of any double curved surface of revolution, an infinite number of tangent planes may be passed. If the axis of revolution is perpendicular to *H,* two such tangent planes may be constructed perpendicular to *V,* having their vertical traces through the vertical projection of the given point, and tangent to the vertical projection of the principal meridian.

PROB. 186. The right section of the outer surface of a main sewer is shown in Fig. 90. A tributary cylindrical sewer of 1′ outside diameter empties into the main sewer at an angle of 60°, the axes of the sewers intersecting in the horizontal plane through the point *a.* To find the curve of intersection.

PROB. 187. To construct a plane through a given line, whose projections are given, and tangent to a surface of revolution by means of an auxiliary hyperboloid of revolution of one nappe.

Analysis. An auxiliary hyperboloid of revolution of one nappe is generated by revolving the given line about the axis of the given surface. The given line is an element of the auxiliary surface, and a plane through this element tangent to the given surface, will also be tangent to the hyperboloid at some point in the element. A plane through the common axis will be a meridian plane of both surfaces, and the tangent plane is perpendicular to the meridian plane at the point of contact on the given surface. Hence, the common meridian plane passed perpendicular to the common tangent plane, contains the points of contact on both surfaces, and intersects the common tangent plane in a line which is tangent to both meridian curves at the points of contact of the tangent plane. This line, tangent to the meridian curves, and the given line, determine the required tangent plane.

If the axis of revolution be taken perpendicular to *H,* a common tangent to the meridian curves parallel to V will give the revolved positions of the points of contact.

PROB. 188. To design the steps and the balustrade and its supports for a spiral staircase, to connect one floor of a building with the floor above. The staircase vestibule is 12 feet square and 21 feet high. The staircase begins at the middle point of one side and winds through an arc of 360° in making the ascent. All of the steps are to run to the side walls. The well-hole in the center is in the form of a right vertical circular cylinder. Each step is to be 7 inches in height, and the shortest one is to be 3 feet long. The balustrade is to be 3 feet high, and runs from a newel post at the bottom to the top.

PROB. 189. To construct two hyperbolic paraboloids tangent to each other along a common element.

Two surfaces are said to *raccord* along a common line when they have a common tangent plane at any point along this line.

Analysis. Any two warped surfaces will raccord along a common element when they have common tangent planes at two points of this element, and also a common plane director. So through any element of a given hyperbolic paraboloid an infinite number of hyperbolic paraboloids may be passed which will be tangent to the first surface all along this element. Let M and N be the directrices, and ab an element of a given hyperbolic paraboloid, a and b being the intersections of the element with M and N respectively. The tangent plane to the given surface at a, which is determined by lines M and ab, and the tangent plane at b, determined by N and ab, may be taken as two common tangent planes. Any line in plane (M, ab) through a, and any other line in plane (N, ab) through b, may be taken as directrices of the required hyperbolic paraboloid. The required tangent hyperbolic paraboloid may then be constructed with these two lines as directrices, and with the same plane director as the first surface.

PROB. 190. To construct two hyperboloids of revolution of one nappe designed to work together tangentially.

Analysis. Any two warped surfaces will raccord along a common line when they have common tangent planes at three points of this line. Let M, N and O be the three rectilinear directrices, and abc a rectilinear element of the given hyperboloid of one nappe, a, b and c being the intersections of the element with the directrices M, N and O respectively. At the point a any line M' in the plane of the lines M and abc may be taken as a directrix for the second surface. Similarly, two other directrices N' and O' may be assumed; and the lines M', N' and O' will be directrices of a hyperboloid of one nappe tangent to the given one along the element abc. If M', N' and O' are all parallel to one plane, the second surface will be a hyperbolic paraboloid.

The lines of the teeth of wheels in hyperboloidal gearing are elements of two tangent hyperbolic paraboloids.

PROB. 191. In a spherical triangle, two sides and an angle opposite one of them are given, to find the other parts.

Analysis. In the spherical pyramid of which the given triangle is the base, the faces and diedral angles are measured respectively by the sides and angles of the given triangle. Therefore, a graphical solution of the spherical triangle will be found when the corresponding triedral angle is constructed. Hence, it is required to construct a triedral angle when two face angles and a diedral angle opposite one of them are given.

In Fig. 91, let the sides of the given triangle be a, b and c, and opposite angles α, β and γ. Likewise, let the faces of the required triedal angle be a, b and c, opposite angles α, β and γ, and edges A, B and C. Let it be assumed that a, b and α are given.

The face b is taken in H, the edge C which is between the given faces being placed perpendicular to GL. Assuming the point o as the vertex of the triedral angle, $A^h oC^h$ represents the face b in its true size.

B_1, drawn at an angle equal to a with C^h is the position of the third edge after it is revolved about C into H.

The face c will lie in a plane whose horizontal trace is A^h and which makes with H the angle α. VD is the vertical trace of this plane.

After the counter-revolution of B_1, the vertical trace of B is found at the intersection of VD, and the arc described by point m. The edge B connects B_{vt} and o.

From the projections of the triedral angle o-ABC, the parts c, γ and β can be readily found.

PROB. 192. In a spherical triangle, two angles and a side opposite one are given, to find the other parts.

PROB. 193. In a spherical triangle, two sides and the included angle are given, to find the other parts.

PROB. 194 In a spherical triangle, the three sides are given, to find the three angles.

60. If the eye of an observer and a point be situated on opposite sides of a surface and a right line be passed connecting them, the intersection of the line with the surface may be called the *linear perspective* of the point. In perspective drawing the position of the eye is called the *point of sight*, the surface is the *picture surface*, and the line is the *visual ray*. The perspective of any right line is the line connecting the perspectives of two points of the given line The perspective of any object is determined by the perspectives of the lines on the body.

PROB. 195. To construct the perspective of a polyedron in the third angle, assuming that V is the picture plane, and taking the point of sight in the fourth angle.

Place the body in the third angle so that its projections may be readily obtained. Assume the point of sight at some distance in front of V to the right or left of the body and a little above it. The perspective of any point of the body may be found by drawing the projections of the visual ray from this point and getting its intersection with V. By connecting the perspectives of points on the body, the perspectives of all the visible edges may be obtained.

PROBLEMS and ILLUSTRATIONS

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DESCRIPTIVE GEOMETRY

PROBLEMS FOR CONSTRUCTION

1. Construct the projections of a point (*a*) in [ ] angle, [ ] inches from *H* and [ ] inches from *V*. (Scale, 1″ = 2″.)

a. *I*–2″–3″.
b. *II*–3″–2″.
c. *III*–4″–4″.
d. *IV*–4″–1″.
e. *I*–0″–0″.
f. *III*–3″–0″.

2. Construct the projections of a line (*S*) in [ ] angle, one end (*a*) of the line [ ] inches from *H* and [ ] inches from *V*, the other end (*b*) [ ] inches from *H* and [ ] inches from *V*. (Scale, 1″ = 3″.)

a. *I*–2″–3″–4″–5″.
b. *II*–0″–4″–1″–3″.
c. *III*–5″–3″–2″–0″.
d. *II*–0″–0″–5″–2″.
e. *I*–4″–2″–5″–3″.
f. *II*–2″–0″–0″–0″.

3. Draw the projections of a line (*S*) in [ ] angle, [ ] to [ ], one end (*a*) of the line [ ] inches from *H* and [ ] inches from *V*, the other end (*b*) [ ] inches from *H* and [ ] inches from *V*. (Scale, 1″ = 4″.)

a. *I*–Per.–*H*–2″–3″–6″–3″.
b. *I*–Per.–*V*–5″–2″–5″–6″.
c. *II*–Par.–*V*–0″–3″–4″–3″.
d. *III*–Par.–*GL*–3″–1″–3″–1″
e. *III*–Par.–*H*–0″–4″–0″–2″.
f. *IV*–Par.–*V*–4″–0″–4″–0″.

4. Construct the traces of a plane (X) in [ ] which is [ ] with H and [ ] with V.

a. *I*–90°–45°.
b. *II*–30°–90°.
c. *III*–90°–90°.
d. *IV*–45°–45°.
e. *I*–30°–60°.
f. *I*– 0°–90°.

5. Given the horizontal projection of a line (M^h) on [ ] and its vertical projection on [ ]; find the traces of the line (M_{ht} and M_{vt}).

a. H_1-V_1.
b. H_2-V_1.
c. H_2-V_2.
d. H_1-V_2.
e. $GL-V_1$.
f. H_2-GL.

6. Given the horizontal trace of a line (S_{ht}) in [ ] and its vertical trace (S_{vt}) in [ ]; find the projections (S^h and S^v) of the line.

a. H_1-V_1.
b. H_1-V_2.
c. H_2-V_2.
d. H_2-V_1.
e. $H_1-\infty$.
f. $\infty-V_2$.

7. Find the length of a line (M) joining a point (a) in [ ] and a point (b) in [ ].

a. *I–II.*
b. *II–III.*
c. *III–III.*
d. *II–IV.*
e. *GL–III.*
f. H_1–*II.*

8. Given the [ ] projection of a line (M) in a plane (Q) whose horizontal trace is in [ ] and whose vertical trace is in [ ]; find the other projection.

a. Ver.–H_1–V_1.
b. Hor.–H_2–V_1.
c. Ver.–H_1–V_2.
d. Hor.–H_2–V_2.
e. Hor.–H_1–V_2.
f. Ver.–H_2–V_2.

10. Find the angle (α) which a line (M) in [ ] makes with [ ] by revolving the line until it becomes parallel with [ ].

a. *I–V–H.*
b. *I–H–V.*
c. *II–V–H.*
d. *III–H–V.*
e. *IV–V–H.*
f. *IV–H–V.*

9. Given the [ ] projection of a point (a) in a plane (Q) whose horizontal trace is in [ ] and whose vertical trace is in [ ]; find the other projection.

a. Hor.–H_1–V_1.
b. Ver. –H_1–V_1.
c. Hor.–H_2–V_1.
d. Ver. –H_1–V_2.
e. Hor.–H_2–V_2.
f. Ver. –H_1–V_2.

11. Assume a point (a) in [ ] lying in a plane (Q) determined by its traces. It is required to find the position of the point after the plane is revolved about its [ ] trace to coincide with [ ].

a. *I*–Ver. – *V.*
b. *II*–Hor. –*H.*
c. *III*–Hor. –*H.*
d. *IV*–Ver. – *V.*
e. V_1–Hor. –*H.*
f. H_2–Ver. – *V.*

12. Revolve a point (a) in [ ] through an angle of [ ] about an axis (M) in [ ] perpendicular to [ ].

a. *I*– 60°– *I–H.*
b. *II*– 45°– *II–V.*
c. *III*–100°–*III–H.*
d. *II*– 30°– *I–H.*
e. *IV*– 60°–*IV–V.*
f. *III*– 45°– *II–V.*

13. Revolve a line (M) in [ ] through an angle of [ ] about an axis (N) in [ ] perpendicular to [ ].

a. *I*–45°– *I–V.*
b. *II*–60°– *II–H.*
c. *III*–75°– *I–H.*
d. *IV*–60°–*IV–V.*
e. *II*–45°– *I–H.*
f. *III*–30°– *II–V.*

14. Cause an oblique line (M) in [ ] by two successive revolutions to become perpendicular to [ ].

a. *I–V.*
b. *II–H.*
c. *II–V.*
d. *III–H.*
e. *III–V.*
f. *IV–H.*

15. Given two lines (M and N) intersecting in [ ]; find the traces of the plane (X) containing them.

a. *I.*
b. *II.*
c. *III.*
d. *IV.*
e. H_1.
f. *GL.*

16. Construct the traces of a plane (X) containing three points, (*a*) in [ ], (*b*) in [ ] and (*c*) in [ ].

a. *I– I– I.*
b. *I– II–III.*
c. *II–III–IV.*
d. *III–III–III.*
e. *GL– I–* V_1.
f. H_2*–GL– II.*

17. Find the angle between two lines (M and N) intersecting in [ ].

a. *I.*
b. *II.*
c. *III.*
d. *IV.*
e. H_1.
f. *GL.*

18. Construct the projections of a line (S) bisecting the angle between two lines (M and N) intersecting in [ ].

a. *I.*
b. *II.*
c. *III.*
d. *IV.*
e. V_1.
f. *GL.*

19. Draw the projections of a line (S) in [ ] which shall make an angle of [ ] with H and an angle of [ ] with V.

a. *I*–30°–45°.
b. *III*–45°–30°.
c. *IV*–20°–60°.
d. *I*–30°–30°.
e. *III*–90°– 0°.
f. *I*–30°–60°.

20. Find the line of intersection (S) of two planes (R and Q), the horizontal traces being in [ ] and the vertical traces in [ ], the horizontal traces intersecting at [ ] and the vertical traces at [ ] from GL.

a. $H_2-V_1-3''-4''$.
b. $H_1-V_2-5''-2''$.
c. $H_1-V_1-2''-In$.
d. $H_1-V_1-0''-0''$.
e. $H_1-V_1-\infty-5''$.
f. $H_1-V_1-In-In$.
g. $H_1-V_1-\infty-\infty$.

21. Find the point (e) in which a line (M) in [ ] intersects a plane (Q), whose horizontal trace is on [ ] at [ ] and whose vertical trace is on [ ] at [ ].

a. $II-H_1-(45^\circ-r)-V_1-(30^\circ-r)$.
b. $I-H_2-(30^\circ-r)-V_1-(60^\circ-r)$.
c. $III-H_2-(60^\circ-l)-V_2-(45^\circ-l)$.
d. $IV-H_1-(60^\circ-l)-V_2-(45^\circ-r)$.
e. $I-H_1-0^\circ-V_1-0^\circ$
f. $I-H_1-90^\circ-V_1-45^\circ$.

22. Find the length of line (S) parallel to GL, 1″ from H and $\frac{1}{2}''$ from V, which is intercepted between planes (Q and R) in Fig. 25.

23. Find the distance from a point (a) in [ ] to a plane (Q), whose horizontal trace is an [ ] and whose vertical trace is an [ ].

a. $I-H_2-V_1$.
b. $II-H_2-V_2$.
c. $III-H_1-V_2$.
d. $IV-H_1-V_2$.
e. $GL-H_1-V_1$.
f. $I-GL-GL$.

24. Project a line (*M*) in [ ] on a plane (*Q*), whose horizontal trace is [ ] and whose vertical trace is [ ].

a. *I*–(30°–*l*)–(45°–*l*).
b. *II*–(45°–*r*)–(45°–*r*).
c. *III*–(60°–*r*)–(60°–*l*).
d. *IV*–(30°–*r*)–(45°–*r*).
e. *III*–(60°–*r*)– 90°.
f. *I*– 90°– 90°.

25. Pass a plane (*X*) through a point (*a*) in [ ], parallel to a plane (*Q*) whose traces are on [ ] and [ ].

a. *I*–H_2–V_1.
b. *II*–H_2–V_1.
c. *III*–H_2–V_2.
d. *IV*–H_1–V_2.
e. V_1–H_1–V_1.
f. *GL*–H_1–V_1.

26. Pass a plane (*X*) through a point (*a*) in [ ] perpendicular to a line (*M*) in [ ].

a. *I*– *II*.
b. *II*– *II*.
c. *III*–*III*.
d. *IV*–*III*.
e. V_1– *I*.
f. *GL*–*IV*.

27. Pass a plane (*X*) through a line (*M*) in [ ] perpendicular to a plane (*P*) whose traces are [ ] and [ ].

a. *I*–(45°–*l*)–(60°–*l*).
b. *II*–(30°–*r*)–(45°–*l*).
c. *III*–(60°–*r*)–(60°–*l*).
d. *IV*–(30°–*l*)–(45°–*l*).
e. *GL*–(30°–*r*)–(30°–*l*).
f. H_2–(45°–*l*)–(60°–*r*).

28. Pass a plane (*X*) through a point (*a*) in [ ] and parallel to two lines (*M* and *N*), (*M*) in [ ] and (*N*) in [ ].

a. *I*– *II*– *I*.
b. *III*–*III*–*III*.
c. *III*–*IV*– *I*.
d. *IV*–*III*– H_1.
e. H_2– *I*– *II*.
f. *GL*– V_2– *I*.

29. Pass a plane (*X*) through a line (*M*) in [ ] parallel to a second given line (*N*) in [ ].

a. *I*– *II*.
b. *III*–*III*.
c. *III*–*IV*.
d. *II*–*IV*.
e. H_1– *II*.
f. *GL*– *I*.

30. The traces of a plane (*Q*) are in [ ] and [ ] and at angles [ ] and [ ]; find the angle which the plane makes with [ ].

a. H_1–V_1–(30°–*r*)–(45°–*r*)–*V*.
b. H_2–V_1–(45°–*r*)–(30°–*r*)–*H*.
c. H_2–V_2–(60°–*l*)–(45°–*l*)–*V*.
d. H_1–V_2–(30°–*l*)–(30°–*r*)–*H*.
e. H_1–V_2–(45°–*l*)– 90°– *V*.
f. H_1–V_1– 90°– 90°– *H*.

31. In Fig. 32, find the angle which plane (*Q*) makes with the profile plane (*P*).

32. The traces are given as in Prob. 29. Find the angle between the traces.

33. Given the [ ] trace of a plane (*Q*), [ ], and the angle between the traces [ ]. Construct the other trace.

a. *H*–(30°–*r*)–45°.
b. *V*–(45°–*r*)–60°.
c. *H*–(60°–*l*)–60°.
d. *V*–(45°–*l*)–75°.
e. *H*–(30°–*r*)–90°.
f. *V*–(90°–*l*)–90°.

34. A corner of a cube is truncated so that the plane of the section makes an angle of [ ] with one face and an angle of [ ] with another face. What angle does it make with the third face?

a. 60°–45°.
b. 45°–60°.
c. 30°–60°.
d. 60°–60°.
e. 90°–90°.
f. 60°–90°.

36. Locate a point (*e*) on a plane given as in Prob. 35, [ ] inches from *H* and [ ] inches from *V*.

a. 2–3.
b. 4–2.
c. 3–0.
d. 0–4.
e. 3–3.
f. 0–0.

35. Construct the traces of a plane (*X*) making angles of [ ] and [ ] with *H* and *V* respectively.

a. 45°–60°.
b. 60°–45°.
c. 90°– 0°.
d. 30°–60°.
e. 90°–90°.
f. 45°–45°.

37. Find the shortest distance from a point (*a*) in [ ] to a line (*M*) in [ ].

a. *I*– *II.*
b. *I*–*III.*
c. *II*–*III.*
d. H_1–*IV.*
e. *GL*–*III.*
f. H_2– V_2.

38. Find the angle between a line (*M*) whose projections are [ ] and [ ] and a plane (*R*) whose traces are [ ] and [ ].

a. (45°–*l*)–(60°–*l*)–(30°–*r*)–(45°–*r*).
b. (30°–*r*)–(45°–*r*)–(30°–*r*)–(60°–*r*).
c. (60°–*l*)–(45°–*r*)–(30°–*r*)–(45°–*l*).
d. 0°– 0°– (45°–*l*)–(30°–*r*).
e. 90°– 90°– (60°–*l*)–(45°–*l*).
f. (30°–*r*)–(45°–*r*)– 0°– 0°.

39. Find the angle between a plane (*R*) whose traces are [ ] and [ ] and a plane (*Q*) whose traces are [ ] and [ ].

a. (45°–*r*)–(30°–*r*)–(45°–*l*)–(30°–*l*).
b. (30°–*r*)–(60°–*l*)–(30°–*l*)–(30°–*l*).
c. (45°–*r*)–(30°–*l*)–(60°–*r*)–(45°–*r*).
d. (45°–*l*)–(30°–*l*)–(60°–*l*)–(45°–*l*).
e. 90°– (45°–*r*)–(45°–*r*)–(60°–*l*).
f. (30°–*l*)– 90°– (60°–*r*)–(45°–*r*).

40. Find the perpendicular distance (*S*) between two lines (*M* and *N*), the projections of (*M*) being [ ] and [ ], and the projections of (*N*) are [ ] and [ ], the lines lying in [ ] and [ ].

a. (45°–*r*)–(60°–*r*)–(60°–*l*)–(30°–*l*)– *I*– *II*.
b. (30°–*l*)–(45°–*r*)–(30°–*r*)–(45°–*r*)– *II*– *II*.
c. (45°–*r*)–(60°–*r*)– 0°– 0°– *III*–*IV*.
d. (30°–*l*)– 0°–)– 45°–*l*)–(60°–*l*)– H_1– *I*.
e. 0°– (45°–*r*)–(30°–*r*)–(45°–*r*)– V_1– *I*.
f. 0°– 0°– (30°–*l*)–(45°–*r*)–*GL*– *I*.

41. A square building is covered by a hip roof. If the hip rafter makes an angle of 30° with *H*, what angle does the roof make with *H?*

42. Find the [ ] projection of a line (*M*) when the other projection is given, the line to pass through a point (*a*) in [ ] and to make an angle of [ ] degrees with [ ].

a. Hor.– *I*–30°–*V*.
b. Ver.– *II*–45°–*H*.
c. Hor.–*III*–60°–*V*.
d. Ver.– *II*–30°–*H*.
e. Hor.–*IV*–45°–*V*.
f. Ver.–*III*–30°–*H*.

43. Fig. 38 represents a cube with holes drilled at (*a*) and (*b*) perpendicular to the faces, with depths of 1″ and $\frac{3}{4}''$ respectively. At what point of the top, and at what angle, must a hole be started to join the ends of the given holes.

44. Revolve a plane (*R*) whose traces are [ ] and [ ] through an angle of [ ] about an axis (*M*) perpendicular to [ ].

a. (30°–*r*)–(60°–*r*)–30°–*H.*
b. (45°–*r*)–(45°–*r*)–60°–*V.*
c. (45°–*l*)–(60°–*l*)–45°–*H.*
d. (30°–*r*)–(45°–*l*)–60°–*V.*
e. (45°–*r*)–(30°–*l*)–30°–*H.*
f. (45°–*r*)–(45°–*l*)–45°–*V.*

45. Find where a line (*M*) whose projections are [ ] and [ ] intersects a plane (*R*) determined by two intersecting lines (*N* and *O*) in [ ] without finding the traces of the plane.

a. (30°–*r*)–(45°–*r*)–*I.*
b. (45°–*r*)–(60°–*r*)–*II.*
c. (45°–*l*)–(45°–*l*)–*III.*
d. (30°–*r*)–(45°–*l*)–*IV.*
e. 90°– 90° –*I.*
f. 0°– 0° –*I.*

46. Find the intersection of the plane (*R*) and a polyedron. Assume that the traces of the plane are [ ] and [ ] and that the polyedron is a [ ].

a. (30°–*r*)–(45°–*r*)–(Tri. Pyr.).
b. (45°–*r*)–(45°–*r*)–(Hex. Pyr.).
c. (30°–*l*)–(45°–*l*)–(Oct. Pyr.).
d. (60°–*l*)–(30°–*l*)–(Tri. Prism).
e. (30°–*r*)–(45°–*l*)–(Sq. Prism).
f. (45°–*l*)–(45°–*r*)–(Hex. Prism).

47. Find the angle between the diagonal of a cube and one of the edges which it cuts.

48. Draw the projections of a regular polygon of [ ] sides situated in a plane (*R*) whose traces are [ ] and [ ].

a. 3–(45°–*r*)–(30°–*r*).
b. 4–(30°–*r*)–(60°–*r*).
c. 5–(30°–*l*)–(45°–*l*).
d. 6– 0° – 0°.
e. 3– 90° – 90°.
f. 4– 0° –In.

49. Find the projections of a regular pyramid whose base of [ ] sides is in a plane (*R*) whose traces are [ ] and [ ].

a. 3–(45°–*r*)–(45°–*r*).
b. 4–(30°–*r*)–(60°–*r*).
c. 5–(45°–*l*)–(30°–*l*).
d. 6–(30°–*l*)–(45°–*r*).
e. 3– 90°– 90°.
f. 4– 0°– 0°.

50. Through a point (*a*) in a line (*M*) which lies in a plane (*R*) whose traces are [ ] and [ ], draw a line (*S*) which shall lie in this plane and make an angle of [ ] with the given line.

a. (30°–*r*)–(45°–*r*)–30°.
b. (30°–*r*)–(60°–*r*)–45°.
c. (45°–*r*)–(30°–*l*)–60°.
d. (30°–*r*)– 90°– 30°.
e. 0° – 0°– 45°.
f. 90° –(45°–*l*)–60°.

51. Find the point (*e*) in a plane (*R*) whose traces are [ ] and [ ] which is at the distances [ ] and [ ] from the traces.

a. (30°–*r*)–(30°–*r*)–2″–3″.
b. (45°–*r*)–(60°–*r*)–3″–4″.
c. 90° –(60°–*l*)–1″–3″.
d. 90° –90° –3″–2″.
e. 0° – 0° –4″–2″.
f. (45°–*r*)–90° –0″–3″.

52. Through a point (*a*) in a plane (*R*) whose traces are [ ] and [ ], construct a line (*S*) which shall lie in the given plane and make an angle of [ ] with [ ].

a. (30°–*r*)–(45°–*r*)–45°–*H.*
b. (45°–*r*)–(30°– *l*)–30°–*V.*
c. (30°–*r*)–(45°– *l*)–60°–*H.*
d. 0° – 0° –30°–*H*
e. 90° –(45°– *l*)–60°–*V.*
f. 90° –(60°–*r*)–45°–*H.*

53. Through a point (*a*) whose distances from *H* and *V*, are [ ] and [ ], construct a line (*S*) parallel to a plane (*R*) whose traces are [ ] and [ ], the line to make an angle of [ ] with [ ].

a. 3″–4″–(30°–*r*)–(45°–*r*)–15°–*H.*
b. 2″–1″–(45°– *l*)–(45°–*l*)–30°–*V.*
c. 3″–2″–(30°– *l*)–(45°–*r*)–20°–*H.*
d. 2″–3″–(45°– *l*)–(60°–*l*)–40°–*H.*
e. 2″–0″–(30°–*r*)–(60°–*r*)–30°–*V.*
f. 0″–0″–(45°–*r*)– 90°– 30°–*H.*

54. Through a point (*a*) in a plane (*R*) making angles of [ ] and [ ] with the planes of projection, draw a line (*S*) lying in the plane and making an angle of [ ] degrees with the [ ] trace.

a. 45°–60°–30°–Ver.
b. 75°–30°–45°–Hor.
c. 60°–30°–60°–Ver.
d. 60°–60°–30°–Hor.
e. 90°–45°–45°–Ver.
f. 80°–60°–60°–Hor.

55. Two parallel horizontal shafts are inclined at 30° to a vertical wall. The first shaft is 4′ above and the second shaft is 5′ below a floor and the distance between the centers of the shafts is 15′. An 8′ pulley on the first shaft just clears the wall by 6″, and drives a 5′ pulley on the second shaft on the opposite side of the wall by an 18″ belt. Show where the wall and the floor must be cut for the belt. [Scale, $\frac{1}{2}''=1'$.]

56. Pass a plane (*X*) through a point (*a*) in [ ] and a line (*M*) in [ ], and revolve the plane about its [ ] trace into [ ], finding the revolved positions of the point and line.

a. *I*– *I*–Hor.–*H*.
b. *I*– *II*–Ver.–*V*.
c. *III*–*III*–Hor.–*H*.
d. *II*– *II*–Ver.–*V*.
e. *III*–*IV*–Hor.–*H*.
f. *I*–*GL*–Ver.–*V*.

57. Pass a plane (*X*) through two parallel lines (*M* and *N*), (*M*) in [ ] and [*N*] in [ ], and revolve the plane about its [ ] trace into [ ], finding the revolved positions of both lines.

a. *I*– *I*–Hor.–*H*.
b. *I*– *II*–Ver.–*V*.
c. *III*–*III*–Hor.–*H*.
d. V_1–*IV*–Ver.–*V*.
e. *GL*– *II*–Hor.–*H*.
f. H_2– V_2–Ver.–*V*.

58. Given the [ ] trace of a plane (*X*) in [ ] and also the projections of one point (*a*) of the plane on [ ] and [ ]; construct the other trace.

a. Hor.–H_1–H_1–V_1.
b. Ver.–V_1–H_2–V_1.
c. Hor.–H_1–H_1–V_2.
d. Ver.–V_2–H_2–V_2.
e.–Hor.–*GL*–H_1–V_1.
f.–Ver.–V_2–H_2–V_1.

59. Given the [ ] trace of a plane (X), [ ], and the angle [ ] between the two traces, to find the other trace.

a. Hor.–(30°–*r*)–60°.
b. Ver.–(45°–*l*)–45°.
c. Hor.–(60°–*r*)–30°.
d. Ver.–(30°–*l*)–60°.
e. Hor.–(45°–*r*)–90°.
f. Ver.–(60°–*l*)–120°.

60. The traces of a plane (P) are given, [ ] and [ ], and the revolved position in [ ] of a point (e) of the plane when the plane has been revolved about its [ ] trace into [ ]; find the projections of the point.

a. (45°–*r*)–(60°–*r*)–H_1–Hor.–H.
b. (30°–*r*)–(45°–*r*)–V_1–Ver.–V.
c. (45°–*r*)–(60°–*l*)–H_2–Hor.–H.
d. (30°–*l*)–(45°–*r*)–V_2–Ver. V.
e. 0°– 0°– H_1–Hor. H.
f. 90°– (45°–*r*) V_1–Ver. V.

61. Given the angle [ ] between the traces of a plane (X), one point (a) in the [ ] trace, and the angle [ ] which the plane makes with [ ]; construct the traces.

a. 60°–Hor.–30°–H.
b. 90°–Ver.–45°–V.
c. 45°–Hor. 60°–H.
d. 121°–Ver. 30°–V.
e. 0°–Hor. 45°–H.
f. 90°–Hor. 60°–H.

62. The lengths of the traces of a plane on H, V, and P, included between the planes, are $2''$, $2\frac{1}{2}''$, and $3''$ respectively. What angles does this plane make with H, V, and P?

63. Construct a line (*S*) through a point (*a*) in [ ] and parallel to two planes (*R* and *Q*) whose horizontal traces intersect in [ ] and whose vertical traces intersect in [ ].

a. I– H_1–V_1.
b. II– H_2–V_1.
c. III– H_2–V_2.
d. GL– H_1–∞.
e. IV–GL–V_1.
f. I– In–V_1.

64. Construct a line (*S*) which shall be parallel to a line (*M*), making angles of [ ] and [ ] with *H* and *V* and which shall intersect two other lines (*N* and *O*), the projections of *N* being [ ] and [ ], and projections of (*O*), [ ] and [ ].

a. 30°–45°–(30°–*r*)–(45°–*r*)–(45°–*l*)–(60°–*r*).
b. 45°–30°–(45°–*l*)–(60°–*l*)–(30°–*r*)–(45°–*r*).
c. 20°–60°–(30°–*r*)–(45°–*l*)–(60°–*r*)–(60°–*l*).
d. 30°–60°– 0°– 0°– (45°–*r*)–(30°–*r*).
e. 60°–30°–(45°–*r*)–(60°–*l*)–(30°–*l*)–(45°–*l*).
f. 90°– 0°– 90°– 90°– (45°–*r*)–(30°–*r*).

65. Find the intersection of three planes (*Q*, *R*, and *L*) whose traces are respectively [ ] and [ ], [ ] and [ ], and [ ] and [ ].

a. (45°–*r*)–(30°–*r*)–(60°–*l*)–(30°–*l*)–(45°–*r*)–(30°–*l*).
b. (30°–*r*)–(60°–*r*)–(45°–*l*)–(30°–*r*)–(60°–*r*)–(30°–*r*).
c. 0°– 0°– 90°– (60°–*r*)–(30°–*r*)–(30°–*r*).
d. 0°– 0°– 0°– 0°– 90°– (60°–*r*).
e. (45°–*l*)–(45°–*l*)–(60°–*r*)–(45°–*r*)–(30°–*l*)– 90°.
f. 90°– (30°–*r*)– 90°– 90°– 0°– 0°.

66. In Fig. 39, a ray of light from (*a*) is reflected from *V* at (*b*) to the profile plane (*P*), and thence to H. Find the points where the reflected ray meets *H* and *P*, and the angles which it makes with *H* and *P*.

67. Cause a plane (*R*) whose traces are [ ] and [ ] by two revolutions to become parallel with a plane (*Q*) whose traces are [ ] and [ ].

a. (30°–*r*)–(60°–*r*)–(30°–*l*)–(45°–*l*).
b. (45°–*l*)–(45°–*l*)–(60°–*r*)–(30°–*r*).
c. (30°–*l*)–(30°–*l*)–(30°–*l*)–(60°–*r*).
d. (60°–*l*)–(60°–*r*)–(30°–*l*)–(30°–*r*).
e. 0°– 0°– (30°–*r*)–(30°–*r*).
f. 90°– (30°–*r*)–(45°–*r*)– 90° *r*.

68. A right pyramid has for its base an equilateral triangle of 3″ on a side. What is the altitude of the pyramid if the diedral angles between its faces are 60°?

69. Construct a line (*S*) which shall be parallel to and lie at a given distance from a plane (*R*) whose traces are [ ] and [ ], and which shall also intersect two lines (*M* and *N*), the projections of (*M*) being [ ] and [ ], and the projections of (*N*) being [ ] and [ ].

a. (30°–*r*)–(45°–*r*)–(45°–*l*)–(45°–*l*)–(30 –*r*)–(60°–*l*).
b. (45°–*l*)–(60°–*l*)–(60 –*r*)–(30°–*r*)– 0°– 0°.
c. (30°–*l*)–(45°–*l*)–(45°–*r*)–(30°–*r*)–(60°–*l*)–(30°–*l*).
d. (45°–*r*)– 90°– 0°– 0°– (45°–*r*)–(30°–*r*).
e. (30°–*r*)–(30°–*r*)–(45°–*r*)– 0°– 90°– (30°–*r*).
f. (45°–*r*)–(30°–*l*)– 0° –(30°–*l*)–(45°–*r*)– 90°.

70. Find the distance between two parallel planes (*R* and *Q*), whose traces are [ ] and [ ].

a. (30°–*r*)–60°–*r*).
b. 90°– (60°–*l*).
c. 0°– 0°.
d. (30°–*l*)–(30°–*l*).
e. (45°–*l*)–(60°–*r*).
f. (60°–*r*)–(60°–*l*).

71. Find the point (*e*) that is [ ] distance from the plane (*R*) whose traces are [ ] and [ ], and [ ] distance from plane (*Q*) whose traces are [ ] and [ ], and [ ] distance from plane (*L*) whose traces are [ ] and [ ].

a. 3″–(45°–*r*)–(60°–*r*)–2″–(30°–*r*)–(45°–*r*)–3″–(45°–*r*)–(45°–*r*).
b. 2″–(30°–*r*)–(30°–*r*)–4″–(60°–*l*)–(60°–*l*)–2″–(30°–*r*) (30°–*l*).
c. 1″–(60°–*r*)–(60°–*r*)–2″– 0°– 0°– 3″– 0°– 0°.
d. 3″– 90°– (45°–*l*)–3°– 0°– 0°– 4″– 90°– (45°–*r*).
e. 2″–(60°–*l*)–(60°–*r*)–3″– 90– (45°–*r*)–5″–(45″–*l*)–(45 –*r*).
f. 4″–(45°–*r*)–(30 –*r*)–2″–(45–*l*)– 90°– 2″– 90°– (60°–*r*).

72. Construct a line (*S*) at a distance [ ] from a plane (*R*) whose traces are [ ] and [ ], and at a distance [ ] from a plane (*Q*) whose traces are [ ] and [ ].

a. 2″–(60°–*l*)–(30°–*l*)–3″–(60°–*r*)–(30°–*r*).
b. 3″–(30°–*r*)–(30°–*r*)–4″–(60°–*r*)–(60°–*r*).
c. 1″–(45°–*l*)–(30°–*r*)–5″–(45°–*r*)–(30°–*l*).
d. 2″–(30°–*r*)–(45°–*l*)–3″–(60°–*l*)–(45°–*r*).
e. 4″–(30°–*r*)–(60°–*r*)–2″– 0°– 0°.
f. 2″–(60°–*r*)–(60°–*l*)–3″–(30°–*r*)–(30°–*l*).

73. Find the plane (*X*) passing through a line (*M*) making the angles [ ] and [ ] with *H* and *V*, and at a distance [ ] from a given point (*a*) in [ ].

a. 30°–45°–3″– *I.*
b. 45°–45°–2″– *II.*
c. 30°–30°–1″–*III.*
d. 60°–20°–3°– *IV.*
e. 30°–60°–4″– *I.*
f. 0°– 0°–2″– *II.*

74. Through a given point (*a*) in [ ] pass a plane (*X*) making the angle [ ] with a line (*M*) whose projections are [ ] and [ ].

a. *I*–30°–(45°–*r*)–(30°–*r*).
b. *II*–45°–(30°–*r*)–(60°–*r*).
c. *III*–60°–(45°–*l*)–(30°–*r*).
d. *IV*–30°– 0°– 0°.
e. *I*–45°– 90°– (30°–*l*).
f. *I*–60°–(45°–*r*)– 90°.

75. Draw a line (*S*) through a point (*a*) in [ ] at a [ ] distance from line (*M*) whose projections are [ ] and [ ] and at a [ ]distance from a line (*N*) whose projections are [ ] and [ ].

a. *I*–2″–(45°–*r*)–(30°–*r*)–3″–(30°–*r*)–(60°–*l*).
b. *II*–3″–(60°–*l*)–(45°–*l*)–2″–(30°–*l*)–(45°–*r*).
c. *III*–4″–(30°–*r*)–(60°–*r*)–3″–(45°–*r*)–(30°–*l*).
d. *IV*–3″–(30°–*l*)–(45°–*l*)–4″–(30°–*r*)–(30°–*l*).
e. *I*–2″– 0°– 0°– 5″–(45°–*r*)–(30°–*l*).
f. *GL*–4″– 90°– (45°–*r*)–2″–(30°–*l*)–(45°–*l*).

76. Find the point (*e*) that is equidistant from four given points, (*a*) being in [ ], (*b*) in [ ], (*c*) in [ ], and (*d*) in [ ].

a. *I*– *I*– *I*– *I*.
b. *I*– *I*– *II*– *III*.
c. *I*– *II*–*III*– *IV*.
d. *I*–*GL*– H_1– *II*.
e. *I*– V_1– H_2–*GL*.
f. *II*–*GL*– H_2– V_2.

77. Pass a plane (*X*) through a line (*M*) in [ ] and equidistant from a point (*a*) in [ ] and another point (*b*) in [ ].

a. I– I–I.
b. I– II–III.
c. II–GL–IV.
d. III– H_1–V_1.
e. GL– I–H_2.
f. H_1–GL–V.

78. Pass a plane (*X*) through a given point (*a*) in [ ], parallel to a line (*M*) in [ ], and perpendicular to a plane (*R*) whose traces are [ ] and [ ].

a. I– I–(45°–*r*)–(60°–*r*).
b. II–II–(30°–*r*)–(60°–*l*).
c. III– I–(45°–*l*)–(30 –*l*).
d. I–H_1– 90°– 45°–*l*.
e. I–V_2– 0°– 0°.
f. GL– I–(45°–*r*)– 90°.

79. Erect a perpendicular (*S*) to a plane (*R*) whose traces are [ ] and [ ] and which shall intersect two lines (*M* and *N*), the projections of (*M*) being [ ] and [ ] and the projections of (*N*), [ ] and [ ].

a. (45°–*r*)–(30°–*r*)–(45°–*l*)–(60°–*l*)–(45°–*r*)–(30°–*l*).
b. (30°–*l*)–(45°–*l*)–(30°–*r*)–(45°–*r*)–(60°–*r*)–(45°–*l*).
c. (30°–*r*)– 90°– (45°–*l*)– 90°– (45°–*r*)–(45°–*l*).
d. (45°–*l*)–(30°–*r*)– 0°– 0°– (45°–*l*)–(60°–*r*).
e. 90°– –(30°–*l*)–(45°–*r*)– 90°– (30°–*r*)–(60°–*r*).
f. (45°–*l*)–(30°–*l*)–(60°–*r*)– 90°– 0°– 0°.

80. Find a point (*e*) in a line (*S*) in [ ] that is at a distance [ ] from a point (*a*) in [ ].

a. I–3″–I.
b. II–2″–II.
c. III–4″–I.
d. H_1–3″–I.
e. GL–3″–II.
f. IV–2″–V_1.

81. A point (*e*) lies at a distance [ ] from a line (*M*) whose projections are [ ] and [ ]. Given one projection of the point on [ ]; find the other projection.

a. 3″–(45°–*r*)–(60°–*r*)–H_1.
b. 2″–(30°–*l*)–(45°–*l*)-V_1.
c. 4″–(30°–*r*)–(30°–*l*)–H_1.
d. 3″–(60°–*r*)–(45°–*l*)–V_1.
e. 1″– 0°– 0°– H_1.
f. 2″– 90°– 90°– V_1.

82. Cause a plane (*R*) whose traces are [ ] and [ ] by two revolutions to take position to make an angle [ ] with *H*, and an angle [ ] with *V*, and also find the revolved position of a polygon in the given plane.

a. (30°–*r*)–(45°–*r*)–60°–45°.
b. (45°–*l*)–(60°–*l*)–45°–45°.
c. (30°–*r*)–(60°–*l*)–45°–60°.
d. 0°– 0°– 75°–30°
e. 90°– (45°–*l*)–60°–60°
f. (60°–*r*)– 90°– 90°– 0°.

83. Draw a line (*S*) through a point (*a*) in [ ] which shall intersect a line (*M*) in [ ] whose projections are [ ] and [ ], and shall make an angle [ ] with a plane (*R*) whose traces are [ ] and [ ].

a. *I*– *I*–(60°–*r*)–(45°–*l*)–30°–(30°–*l*)–(30°–*l*.
b. *I*– *II*–(60°–*l*)–(60°–*r*)–45°–(30°–*r*)–(45°–*r*).
c. *II*–*III*–(60°–*r*)–(60°–*l*)–60°–(30°–*l*)–(30°–*r*).
d. *I*–*GL*–(45°–*l*)–(45°–*l*)–30°– 0°– 0°.
e. *I*– *I*– 0°– 0°– 45°– 0°– 0°.
f *V*– *I*–(60°–*l*) (60°–*r*)–60°– 90°– 90°.

84. Given the line of intersection (M) of two planes (X and Y) in [ ], the angle [ ] between them and also a point (a) in [ ] which is a point in the [ ] trace of one of the planes; construct the traces of the planes.

a. *I*–60°– H_1–Hor.
b. *II*–90°– V_1–Ver.
c. *III*–45°– H_2–Hor.
d. *IV*–60°– V_2–Ver.
e. H_1–30°– V_1–Ver.
f. *I*–60°–*GL*–Hor.

85. Construct the traces of a plane (X) bisecting the angle between a plane (R) whose traces are [ ] and [ ], and a plane (Q) whose traces are [ ] and [ ].

a. (30°–*r*)–(45°–*r*)–(45°–*r*)–(60°–*r*).
b. (45°–*r*)–(45°–*r*)–(60°–*l*)–(30°–*l*).
c. (45°–*l*)–(45°–*r*)–(60°–*r*)–(30°–*l*).
d. 0°– 0°– 90°– (45°–*l*).
e 90 – (30°–*r*)–(45°–*r*)–(60°–*l*).
f. 0°– 0°– 0°– 0°.

86. Construct the traces of a plane (X) passing through a line (M) in [ ] and making an angle [ ] with a plane (R) whose traces are [ ] and [ ].

a. *I*–30°–(45°–*r*)–(60°–*r*).
b. *II*–60°–(45°–*l*)–(45°–*l*).
c. *III*–45°–(45°–*r*)–(60°–*l*).
d. *IV*–30°– 0°– 0°.
e. *I*–60°– 90°– (45°–*l*).
f. *I*–45°– 90°– 90°.

87. Find the point (e) of a line in [ ] which is equidistant from two planes (R and Q) whose traces are [ ] and [ ], and [ ] and [ ].

a. *I*–(45°–*r*)–(60°–*r*)–(30°–*l*)–(45°–*l*).
b. *II*–(30°–*r*)–(45°–*r*)–(45°–*r*)–(60°–*l*).
c. *III*–(45°–*r*)–(60°–*l*)–(30°–*r*)– 90°.
d. *IV*– 0°– 0°– (45°–*r*)–(30°–*l*).
e. *I*– 0°– 0°– 0°– 0°.
f. *GL*–(45°–*r*)–(60°–*l*)–(30°–*r*)–(45°–*r*).

88. Pass a plane (X) through a line (M) in [ ] which shall make an angle of [ ] with a plane (R) whose traces are [ ] and [ ].

a. *I*–30°–(45°–*r*)–(60°–*r*).
b. *II*–60°–(60°–*l*)–(45°–*l*).
c. *III*–45°–(45°–*l*)–(45°–*r*).
d. *IV*–30°– 0°– 0°.
e. *I*–60°– 90°– 90°.
f. *I*–45°– 90°– (30°–*l*).

89. Draw the traces of a plane (X) which shall lie at distances [ ], [ ], and [ ] from points (a, b, and c) in [ ], [ ], and [ ] respectively.

a. 1″–2″–3″–*I*–*I*–*I*.
b. 2″–3″–4″–*I*–*II*–*III* .
c. 2″–1″–3″–*II*–*III*–*IV*.
d. 3″–4″–1″–*GL*–*I*–H_1.
e. 2″–1″–4″–V_1–*GL*–H_1.
f. 4″–3″–2″–*GL*–*GL*–V_2.

90. Draw the traces of a plane (*X*) perpendicular to [ ], and at an angle of [ ] with a line (*M*), whose projections are [ ] and [ ].

a. *H*–30°–(45°–*r*)–(30°–*r*).
b. *V*–60°–(30°–*l*)–(45°–*l*).
c. *H*–45°–(45°–*r*)–(60°–*r*).
d. *V*–30°– 0°– 0°.
e. *H*–60°– 90°– 90°.
f. *V*–45°–(30°–*l*)–(45°–*l*).

91. Draw the traces of a plane (*X*), which is equidistant from two lines (*M* and *N*), the projections of (*M*), [ ] and [ ], and of (*N*), [ ] and [ ].

a. (30°–*r*)–(45°–*r*)–(60°–*l*)–(45°–*l*).
b. (30°–*l*)–(45°–*l*)–(45°–*r*)–(30°–*l*).
c. 0°– 0°– (30°–*r*)–(45°–*r*).
d. *GL*– *GL*– (45°–*r*)–(30°–*r*).
e. 90°– 90°– (30°–*r*)–(60°–*l*).
f. (45°–*r*)–(60°–*r*)–(30°–*l*)–(45°–*r*).

92. Draw a line (*S*) through a given point (*a*) in [ ], at a distance of [ ] from a point (*b*) in [ ], and which shall intersect a line (*M*) in [ ].

a. *I*–3″– *I*–*I*.
b. *II*–2″– *II*–*II*.
c. *III*–3″– *I*–*II*.
d. *IV*–1″– *I*–*III*.
e. *GL*–4″–H_1–*I*.
f. *I*–2″– V_1–H_2.

93. Construct the shortest line (*S*) which can be drawn parallel to a plane (*R*) whose traces are [ ] and [ ] and terminating in two lines (*M* and *N*), the projections of (*M*) being [ ] and [ ], and the projections of (*N*) being [ ] and [ ].

a. (30°–*r*)–(45°–*r*)–(30°–*r*)–(45°–*r*)–(60°–*l*)–(45°–*r*).
b. (45°–*l*)–(60°–*r*)–(30°–*l*)–(45°–*l*)–(60°–*r*)–(30°–*r*).
c. 90°– (45°–*l*)–(45°–*l*)–(60°–*r*)– 0°– 0°.
d. (45°–*r*)– 90°– (30°–*l*)–(60°–*l*)–(45°–*r*)–(45°–*r*).
e. 0°– 0°– (30°–*l*)–(45°–*l*)–(60°–*r*)–(60°–*l*).
f. (30°–*r*)–(45°–*r*)–(60°–*l*)–(45°–*l*)– 90°– 90°.

94. Construct the traces of a plane (X) which shall contain a line (M) in [ ], one of the traces making an angle of [ ] with the given line.

- *a.* *I*–30°.
- *b.* *II*–60°.
- *c.* *III*–45°.
- *d.* *IV*–30°.
- *e.* H_1–45°.
- *f.* V_2–60°.

95. Given the focus and directrix of a parabola; construct the curve and draw a tangent to the curve at a point on it.

96. Given the foci of an ellipse and the sum of the distances from the foci to a point of the curve; construct the curve, and draw a tangent to the curve at any point on it.

97. Given the two axes of an ellipse; construct the curve by concentric circles on the axes, and draw a tangent to the curve from a point without the curve.

98. Given the foci and the transverse axis of an hyperbola; construct the curve, and draw a tangent to the curve at any point on it.

99. Construct a cycloid, and draw a tangent to the curve at any point on it.

100. Construct an epicycloid, and draw a tangent to the curve at any point on it.

101. Construct the hypocycloid, and a tangent to the curve at any point on it.

102. Construct an involute of a circle, and a tangent to the curve at any point on it.

103. Construct the conchord of Nicomedes.

104. Construct the logarithmic spiral.

105. Construct the spiral of Archimedes.

106. Construct a sinusoid and a tangent to the curve at any point on the curve.

107. Construct Schiele's anti-friction curve.

108. Construct the helix and a tangent to the curve at any point, and also develop the curve on a plane surface.

109. Find the projections of a circle lying in a plane whose traces are [ ] and [ ].

a. (30°–*r*,)–90°.
b. 90°–(60°–*r*.)
c. (60°–*l*)–(30°–*r*).
d. (45°–*l*)–(30°–*r*).
e. (45°–*r*)–(45°–*r*).
f. (30°–*r*)–(60°–*l*).

110. Find in Fig. 59 where the irregular curve *a b* intersects the planes of projection and the plane *R*.

111. Draw the projections of a helix having its axis [ ] to *H* and [ ] to *V*.

a. 45°– 0°.
b. 0°–45°.
c. 0°–60°.
d. 30°–30°.
e. 30°–45°.
f. 45°–45°.

112. Draw the projections of a cylinder, and assume a point on the surface.

113. Pass a plane tangent to a cylinder whose axis makes the angle [ ] with *H* and angle [ ] with *V*.

a. 30°–45°.
b. 45°–30°.
c. 60°–30°.
d. 30°–30°.
e. 45°–45°.
f. 30°–20°.

114. Construct the projections of a right circular cylinder having its axis parallel to *GL*, find a point on the surface, and pass a plane tangent to the surface at this point.

115. A cylinder, whose right section is a circle of $\frac{3}{4}''$ radius, has its axis inclined at 30° to *H* and 45° to *V*. Find the intersections with *H*, *V* and *P*.

116. Pass a plane through a point without a cylinder tangent to the surface, when the projections of the axis are [ ] and [ ].

a. (45°–*r*)–(45°–*r*).
b. (30°–*r*)–(45°–*r*).
c. (45°–*l*)–(60°–*l*).
d. (30°–*l*)–(45°–*r*).
e. 90°– 90°.
f. 0°, 0°.

117. Pass a plane tangent to a cylinder, the projections of the axis being [ ] and [ ], and parallel to a right line whose projections are [ ] and [ ].

a. (45°–*r*)–(45°–*r*)–(45°–*r*)–(60°–*l*).
b. (60°–*r*)–(30°–*r*)–(45°–*r*)–(30°–*l*).
c. (45°–*l*)–(30°–*l*)–(90°– 90°.
d. 0°– (45°–*l*)– 0°– 0°.
e. 90°– 0°– (30°–*r*)–(45°–*r*).
f. 0°– 0°– (45°–*l*)–(60°–*r*).

118. Construct the projections of an oblique cone, and assume a point of the surface.

119. Pass a plane tangent to an oblique [ ] cone at a point on the surface when the projections of the axes are [ ] and [ ].

a. circular, –(30°–*r*)–(45°–*r*).
b. elliptical, –(45°–*r*)–(30°–*r*).
c. parabolic, –(60°–*l*)–(45°–*l*).
d. hyperbolic, –(30°–*l*)–(30°–*l*).
e. circular, – 0°– (45″–*l*).
f. elliptical, –(30°–*l*)– 0°.

120. Pass a plane through a given point without a cone tangent to the surface, when the projections of the axis are [ ] and [ ].

a. (30°–*r*)–(45°–*r*).
b. (45°–*r*)–(60°–*r*).
c. (30°–*l*)–(45°–*l*).
d. 0°– (45°–*l*).
e. 0°– 90°.
f. 0°– 0°.

121. Pass a plane tangent to a cone and parallel to a given right line when the projections of the axis are [ ] and [ ].

a. (45°–*r*)–(60°–*r*).
b. (30°–*r*)–(45°–*r*).
c. (45°–*l*)–(45°–*r*).
d. 90°– 0°.
e. 0°– 0°.
f. 90°– 90°.

122. A right circular cone, the radius of the base being 1″ and altitude 2″, stands on *H*. If light comes from a point $\frac{1}{2}''$ above the vertex and 1″ from the axis, find the line of shade on the cone and the shadow on *H*.

123. Construct the traces of a plane, which shall contain a line making angles of [ ] and [ ] with *H* and *V* respectively, and which shall make an angle of [ ] with [ ].

a. 30°–45°–60°–*H*.
b. 45°–30°–45°–*V*.
c. 45°–20°–45°–*H*.
d. 30°–60°–60°–*V*.
e. 45°–30°–90°–*H*.
f. 0°– 0°–30°– *V*.

124. Construct the projections of a developable helicoid, and assume a point on the surface.

125. Construct the traces of a plane tangent to a developable helicoid at any point on the surface.

126. Pass a plane through a point without a developable helicoid tangent to the surface.

127. Pass a plane tangent to a developable helicoid and parallel to a given right line.

128. Construct the projections of a sphere, assume a point on the surface, and construct a plane tangent to the surface at a point of the surface.

129. Given a sphere whose diameter is 3″, and whose center is in *H*, 1″ from *V*. Pass a plane tangent to the sphere and inclined 30° to *H* and 45° to *V*, and find the point of tangency.

130. Construct the projections of an ellipsoid, assume a point on the surface, and pass a plane tangent to the surface at a point of the surface.

131. Construct the projections of a paraboloid, assume a point on the surface, and pass a plane tangent to the surface at a point of the surface.

132. Construct the projections of the hyperboloid of revolution of two nappes, assume a point on the surface, and pass a plane tangent to the surface.

133. Construct the projections of an annular torus, assume a point on the surface, and construct a plane tangent to the torus at a point on the surface.

134. Pass the surface of a sphere through four points in [ ], [ ], [ ], and [ ], and find the length of the radius.

a. *I– I– I– I.*
b. *I– II–III–IV.*
c. *II– II– II– II.*
d. *I– II–III–III.*
e. *I–III–IV–IV.*
f. *III–III–IV–IV.*

135. Pass a plane through a line whose projections are [ ] and [ ] and tangent to a sphere whose center is in [ ].

a. (30°–*r*)–(45°–*r*)–*I.*
b. (30°–*l*)–(45°–*r*)–*II.*
c. (45°–*r*)–(60°–*l*)–*III.*
d. (30°–*l*)–(45°–*r*)–*IV.*
e. (90°– 90° –*I.*
f. 0°– 0° –*I.*

136. Find the intersection of a right circular cylinder by a plane whose traces are [ ] and [ ], draw a tangent to the curve of intersection, show the curve in its true dimensions, and develop the cylinder and the curve of intersection.

a. (30°–*r*)–(45°–*r*).
b. (45°–*r*)–(60°–*r*).
c. (30°–*r*)–(45°–*l*).
d. (45°–*r*)–(60°–*l*).
e. (30°–*l*)–(60°–*l*).
f. (45°–*l*)–(60°–*r*).

137. Find the intersection of a plane whose traces are [ ] and [ ] with an oblique cylinder, draw a tangent to the curve of intersection, and show the curve in its true dimensions.

a. 90°– (45°–*r*).
b. (30°–*l*)–90°.
c. (30°–*l*)–(45°–*r*).
d. (45°–*r*)–(60°–*r*).
e. (30°–*l*)–(45°–*l*).
f. 0°– 0°.

138. Find the intersection of a right circular cone by a plane whose traces are [ ] and [ ], draw a tangent to the curve of intersection, and develop the cone and the curve.

a. (45°–*r*)– 90°.
b. 90°– (30°–*r*).
c. (45°–*r*)–(30°–*r*).
d. (30°–*l*)–(60°–*l*).
e. (45°–*r*)–(45°–*l*).
f. 0°– 0°.

139. Pass a plane so as to cut a [ ] from a right circular cone, and show the conic section in its true dimensions.

a. circle.
b. ellipse.
c. parabola.
d. hyperbola.

140. Find the intersection of an oblique plane with a cone whose base is in [ ] having the projections of its axis [ ] and [ ].

a. H_1–(30°–*r*)–(45°–*r*).
b. V_1–(45°–*r*)–(60°–*r*).
c. H_2–(30°–*r*)–(60°–*l*).
d. V_2–(45°–*l*)–(45°–*l*).
e. H_1– 90°– 90°.
f. V_1– 90°– 90°.

141. Find the intersection of a plane with a [ ], and draw a tangent to the curve of intersection.

a. annular torus.
b. paraboloid.
c. sphere.
d. paraboloid.
e. ellipsoid.
f. hyperboloid.

142. Find the intersection of a triangular pyramid with a sphere.

143. Find the intersection of a cone, the projections of its axis being [ ] and [ ], and a cylinder, the projections of whose axis are [ ] and [ ].

a. (30°–*r*)–(45°–*r*)–(60°–*l*)–(30°–*l*).
b. (45°–*r*)–(30°–*l*)–(45°–*r*)–(45°–*r*).
c. (45°–*l*)–(45°–*l*)–(30°–*r*)–(45°–*r*).
d. (60°–*l*)–(45°–*l*)–(45°–*l*)–(30°–*r*).
e. (30°–*l*)–(45°–*r*)–(30°–*r*)–(60°–*l*).
f. 0° –(30°–*l*)–(45°–*r*)–(30°–*l*).

144. Let the base of a cylinder be in *H*, and the base of an intersecting cone in *V*. Find the curve of intersection, and a tangent to it.

145. Find the intersections of two oblique cylinders, and draw a tangent to a curve of intersection at any point.

146. Find the intersection of two cones, and draw a tangent to the curve of intersection at any point.

147. Find the intersection of two surfaces of revolution whose axes are in the same plane, and draw a tangent to the curve of intersection at any point.

148. Given two spheres, diameters 2″ and 3″, with centers 2″ apart on a line making angles of 30° and 45° with H and V. Find the traces of the plane and the radius of the circle in which the spheres intersect.

149. Find the intersections of a cylinder and sphere whose projections are given in Fig. 75.

150. Find the intersection of a cone and a sphere when the vertex of the cone is at the center of the sphere, and draw a tangent to the curve of intersection.

151. In Fig. 76 are shown the projections of a hexagonal washer. M is the axis of a cylindrical hole of $\frac{1}{4}''$ radius which is bored through the washer. Find the intersections.

152. A "V" threaded screw is generated by revolving an equilateral triangle about an axis in the same plane, one side of the triangle being parallel to this axis. The side of the triangle is $\frac{1}{2}''$, and the screw base, 2″ pitch and 3″ outside diameter. Find the intersection of a plane parallel to the axis and 1″ from it.

153. Given the plane director and two rectilinear directrices of a hyperbolic paraboloid; construct elements of the surface, and assume a point on the surface.

154. Construct elements of the hyperbolic paraboloid by proportional division of the directrices, and draw an element parallel to a given right line.

155. Pass a plane tangent to a hyperbolic paraboloid at a given point on the surface.

156. Construct elements of the hyperboloid of one sheet by revolving a right line about another right line not in the same plane, and draw elements of both generations.

157. Construct a plane tangent to a hyperboloid of revolution of one nappe at any point of the surface, and find a meridian curve of the surface.

158. Construct elements of a helicoid, and pass a plane tangent to the surface at any point.

159. Construct elements of a warped surface with any three rectilinear directrices.

160. Construct elements of a warped surface with three linear directrices, [ ] of them being curved.

a. one. *b.* two. *c.* three.

161. Construct a conoid and a plane tangent to the surface at any point.

162. Construct elements of a cylindroid.

163. Construct elements of a warped arch.

164. Construct an elliptical hyperboloid, and pass a plane tangent to the surface at any point.

165. Construct the projections of a serpentine, and pass a plane tangent to the surface at any point.

166. Construct a right hyperbolic paraboloid and a line of striction.

167. Construct projections of a developable helicoid, the projections of whose axes make angles of [ ] and [ ] with *GL*.

a. 0°– (45°–*r*). *b.* (30°–*r*)– 0°. *c.* (30°–*r*)–(60°–*r*).
d. (45°–*r*)–(60°–*l*). *e.* (30°–*l*)–(30°–*l*). *f.* 90°– 90°.

168. Inscribe a sphere in a given triangular pyramid.

169. Pass a plane through the point *a* in Fig. 86, tangent to the spheres whose centers are *b* and *c*.

170. Pass a plane so as to cut a [ ] from a hyperbolic paraboloid.

a. parabola. *b.* ellipse.

171. Find the intersection of an oblique plane with an oblique cone, and develop the cone and the curve of intersection.

172. Pass planes so as to cut sub-contrary sections from an oblique [ ] with a circular base.

a. cylinder. *b.* cone.

173. Find the intersection of a plane and a helicoid, and draw a tangent to the curve of intersection.

174. Fig. 87 shows the projections of a bolt-head, partly square and partly conical, and Fig. 88 shows a hexagonal bolt-head with its end turned to a spherical shape; find the curve e^vf^v in each figure.

175. Find the intersection of a plane and a [ ].

a. developable helicoid.
b. right conoid.
c. hyperbolic paraboloid.
e. hyperboloid of one nappe.
f. warped arch.

176. Find where a line whose projections are [ ] and [ ] intersects a [ ].

a. (30°–*r*)–(45°–*r*)–sphere.
b. (30°–*r*)–(60°–*r*)–cone.
c. (45°–*l*)–(30°–*r*)–annular torus.
d. (30°–*l*)–(45°–*l*)–helicoid.
e. (45°–*r*)–(45°–*r*)–serpentine.
f. (30°–*r*)–(30°–*r*)–ellipsoid.

177. Find the intersection of an oblique circular cylinder, the projections of whose axis are [ ] and [ ], with a [ ].

a. 0°– 90°–conoid.
b. 30°– 0°– hyberbolic parab.
c. (60°–*r*)–(30°–*r*)–devel. helicoid.
d. (30°–*l*)–(45°–*l*)–cylindroid.
e. (45°–*l*)–(30°–*l*)–hypone sheet.
f. (30°–*l*)–(45°–*r*)–warped arch.

178. Find the intersection of an oblique [ ] prism with a [ ].

a. triangular–sphere.
b. quadrangular–cone.
c. pentangular–cylinder.
d. hexagonal–ellipsoid.
e. triangular–annular torus.
f. quadrangular–hyperboloid of revolution.

179. Construct the projections of a right groined vault whose covering is one foot in thickness, when the cylindrical axes are in the same plane parallel to H and equally inclined to V.

180. In Fig. 89 are shown the plan and the front and end elevations of a round rod turned in an ordinary lathe from a rectangular bar. It is required to find h^h–g^h and e^v–f^v, the projections of the lines of visible contour of this surface of revolution connecting the round rod and rectangular bar.

181. Pass a plane tangent to a helicoid and parallel to a given right line.

182. Pass a plane tangent to a helicoid, and perpendicular to a given line.

183. Pass a plane tangent to two oblique cones having a common vertex, the base of one being in [ ] and the base of the other in [ ].

a. H_1-H_1.
b. V_1-V_1.
c. H_1-H_2.
d. V_1-V_1.
e. H_1-H_1.
f. H_2-V_1.

184. Construct the projections of one coil of a screw conveyor of the form of a developable helicoid. The diameter of the inner helix being 6″, of the outer 12″, pitch 20″.

185. Pass a plane through a point without the surface tangent to a [ ].

a. sphere.
b. prolate ellipsoid
c. oblate ellipsoid.
d. paraboloid.
e. hyperboloid of rev.
f. annular torus.

186. The right section of the outer surface of a main sewer is shown in Fig. 90. A tributary cylindrical sewer of 1′ outside diameter empties into the main sewer at an angle of 60°, the axes of the sewers intersecting in the horizontal plane through the point *a*. Find the curve of intersection.

187. Pass a plane through a line whose projections are [ ] and [ ] and tangent to a [ ], by means of an auxiliary hyperboloid of revolution of one nappe.

a. (30°–*r*)–(30°–*r*)–sphere.
b. (60°–*r*)–(45°–*r*)–prolate ellipsoid.
c. (60°–*l*)–(60°–*r*)–oblate ellipsoid.
d. (45°–*r*)–(30°–*l*)–paraboloid.
e. (30°–*r*)–(30°–*l*)–cylindroid.
f. (45°–*l*)–(30°–*l*)–annular torus.

188. Design the steps and the balustrade and its supports for a spiral staircase to connect one floor of a building with the floor above. The staircase vestibule is 12 feet square and 21 feet high. The staircase begins at the middle point of one side, and winds through an arc of 300° in making the ascent. All of the steps are to run to the side walls. The well-hole in the center is in the form of a right vertical circular cylinder. Each step is to be 7 inches in height, and the shortest one is to be 3 feet long. The balustrade is to be 3 feet high, and runs from a newel post at the bottom to the top.

189. Construct two hyperbolic paraboloids tangent to each other along a common element.

190. Construct two hyperboloids of revolution of one nappe designed to work together tangentially as in hyperboloidal gearing.

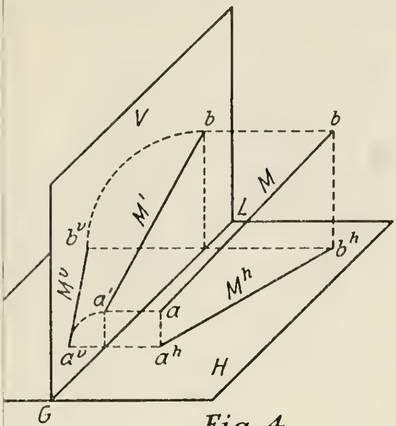
191. In a spherical triangle, two sides and an angle opposite one of them are given; find the other parts.

192. In a spherical triangle, two angles and a side opposite one are given; find the other parts.

193. In a spherical triangle, two sides and the included angle are given; find the other parts.

194. In a spherical triangle, the three sides are given; find the three angles.

Figs. 1 to 12



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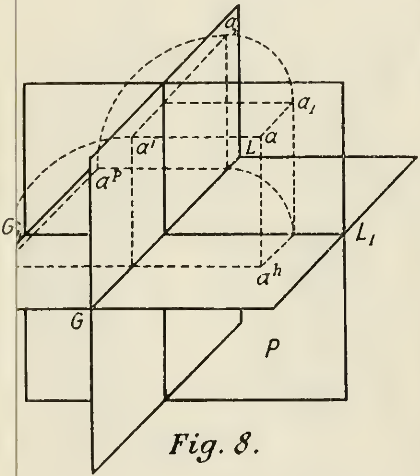

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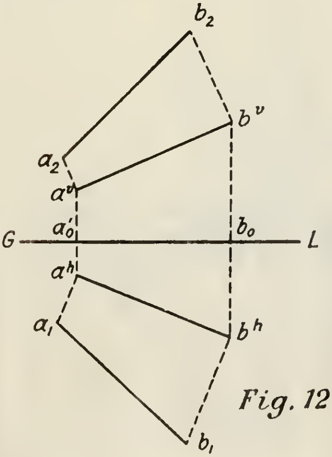

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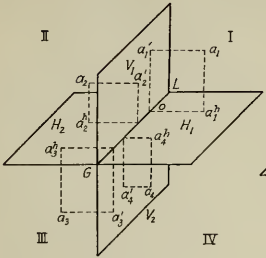

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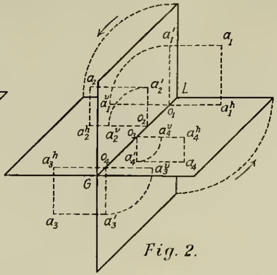

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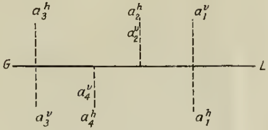

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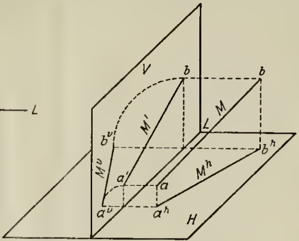

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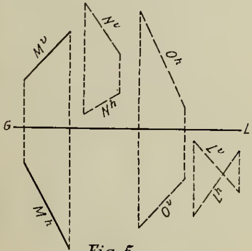

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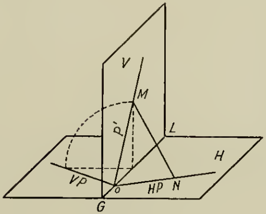

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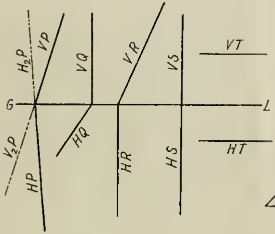

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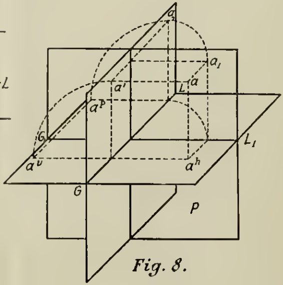

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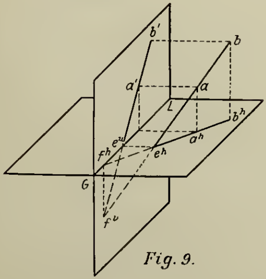

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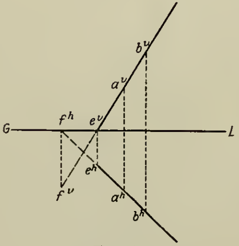

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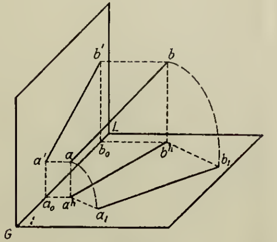

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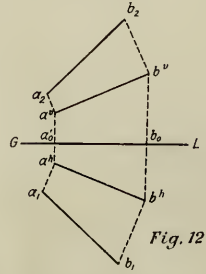
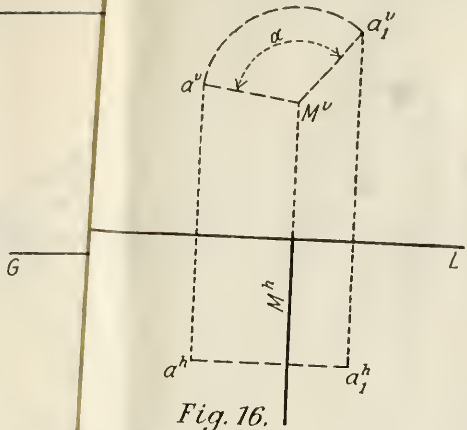


Fig. 12.

Figs. 13 to 24



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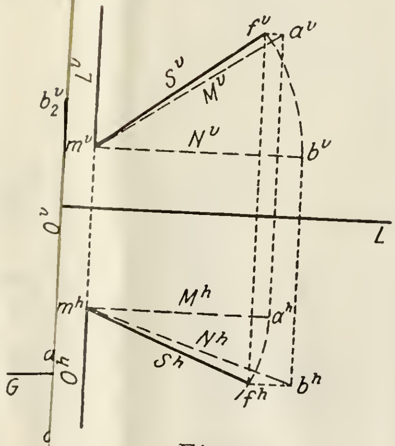

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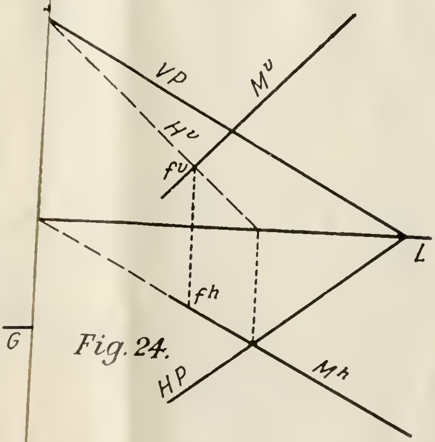

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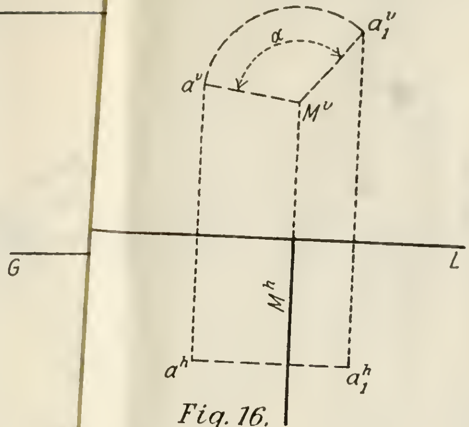

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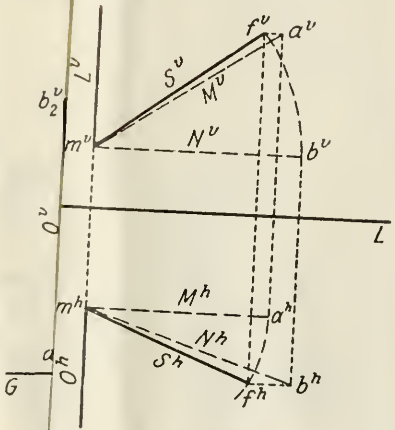

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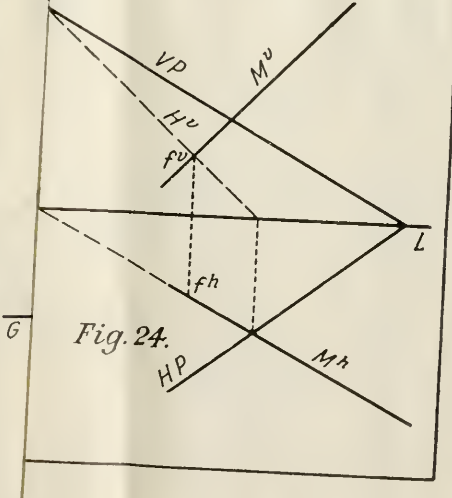

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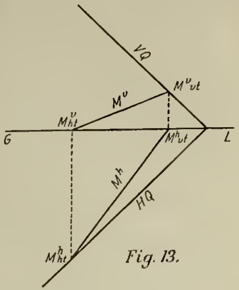

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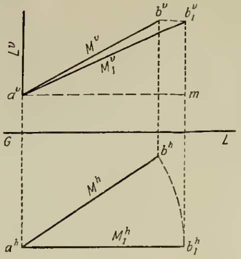

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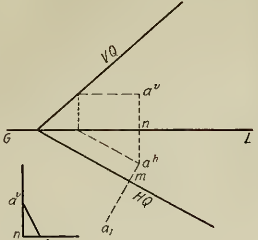

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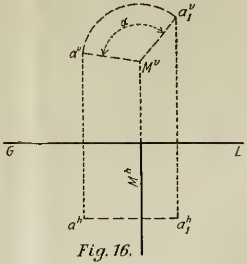

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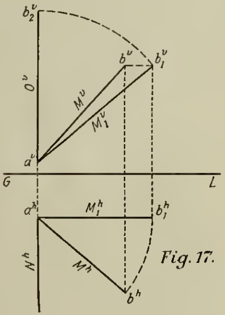

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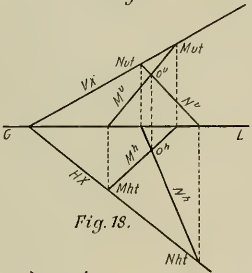

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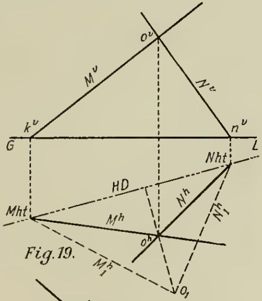

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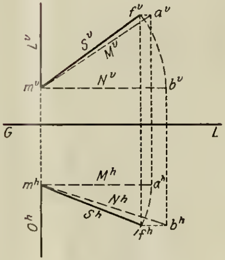

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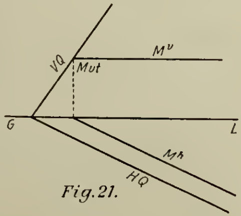

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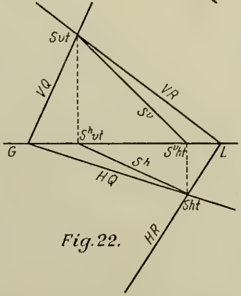

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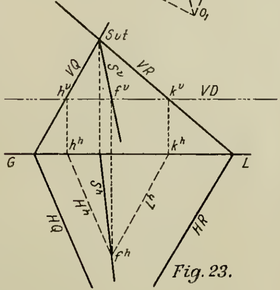

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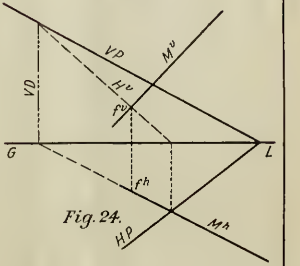

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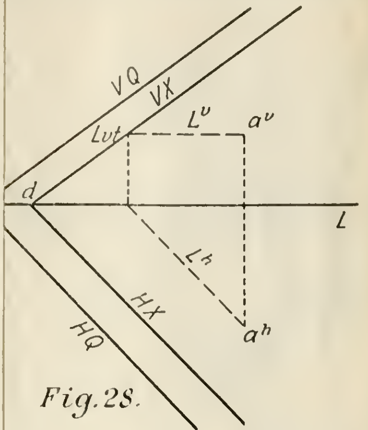
Figs. 25 to 36



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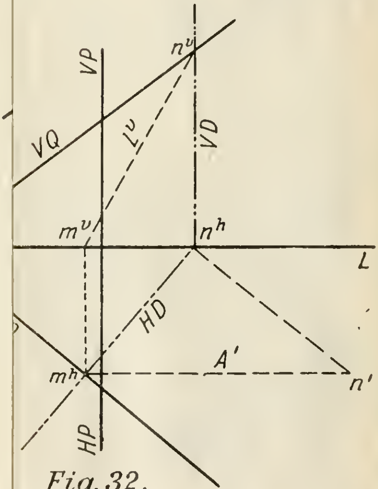

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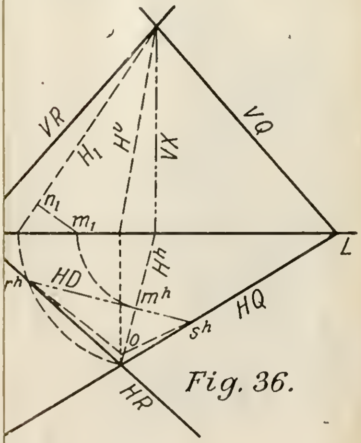

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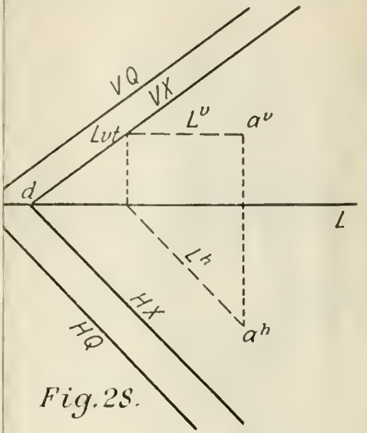

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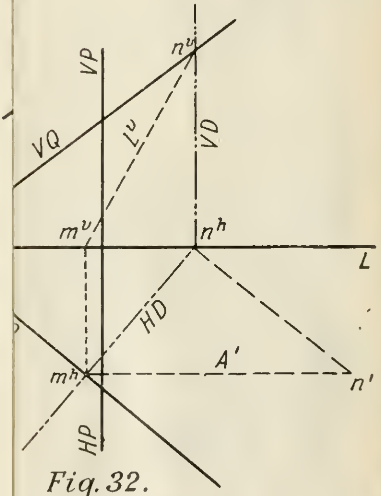

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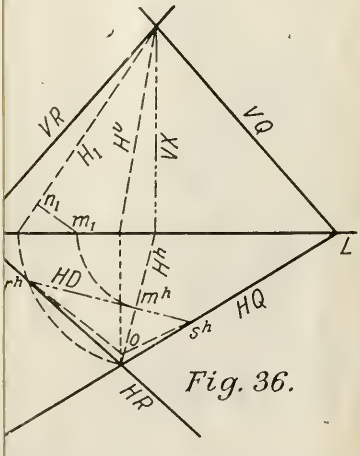

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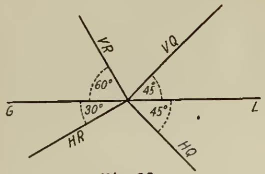

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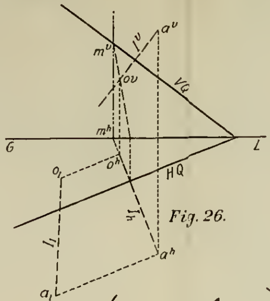

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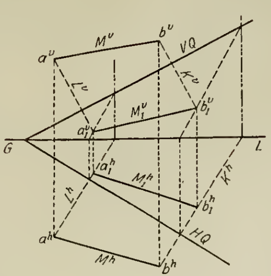

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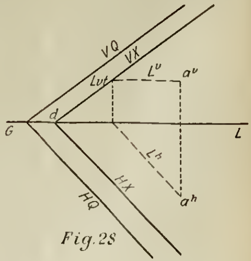

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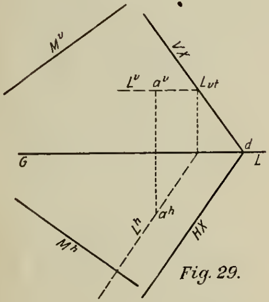

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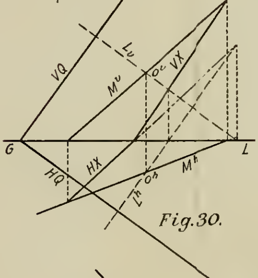

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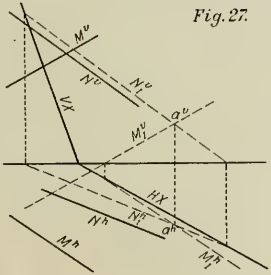

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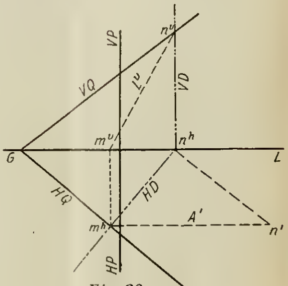

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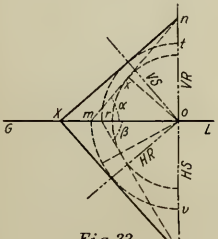

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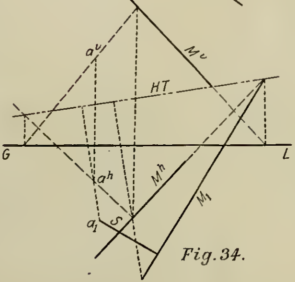

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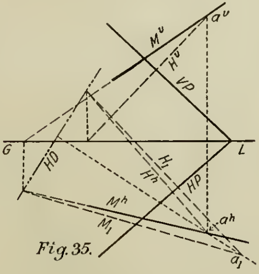

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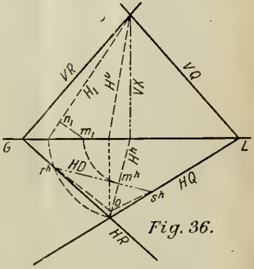

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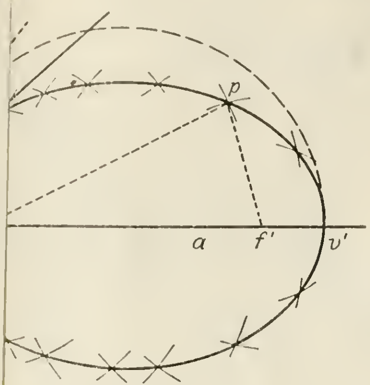
Figs. 37 to 47



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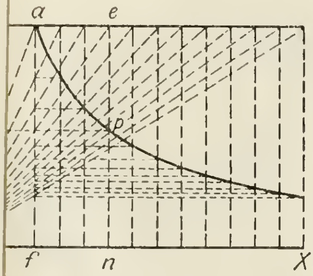

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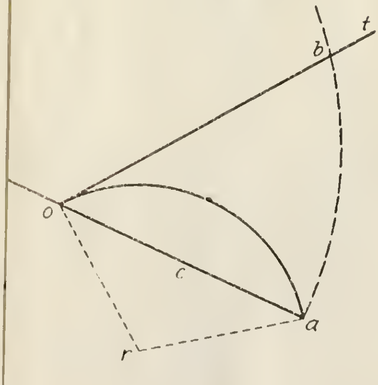

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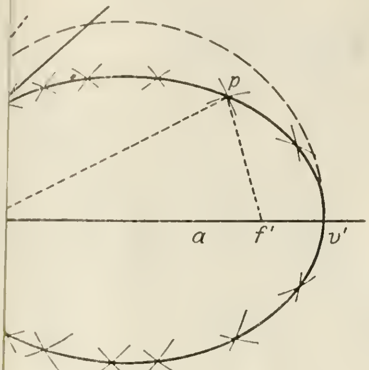
Figs. 37 to 47



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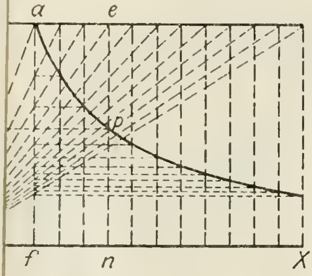

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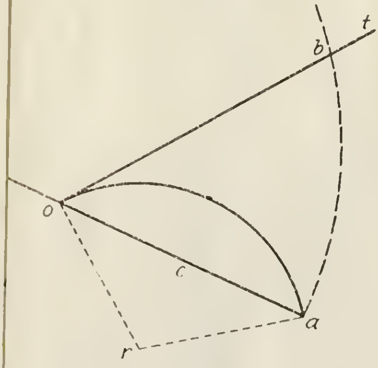

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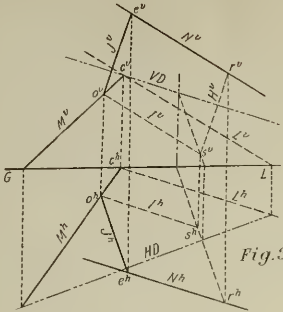

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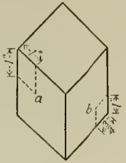

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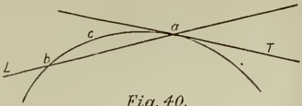

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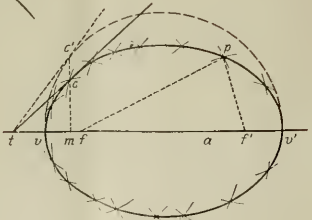

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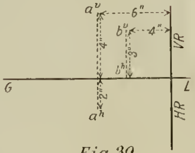

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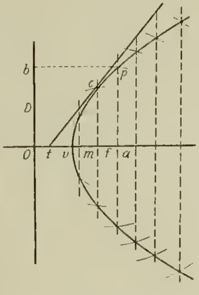

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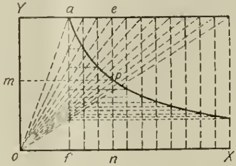

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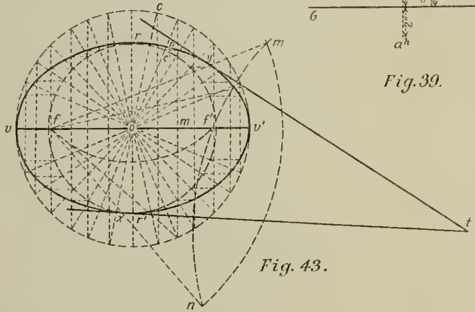

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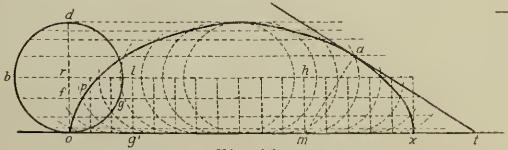

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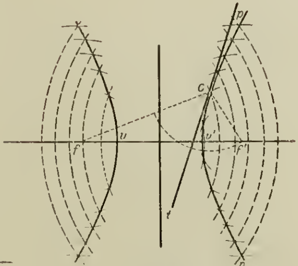

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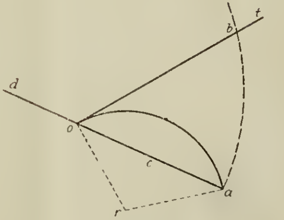

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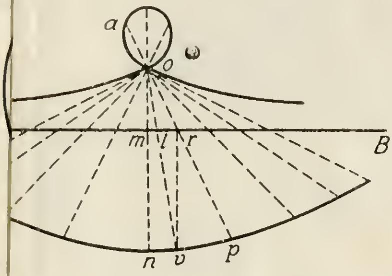
Figs. 48 to 58



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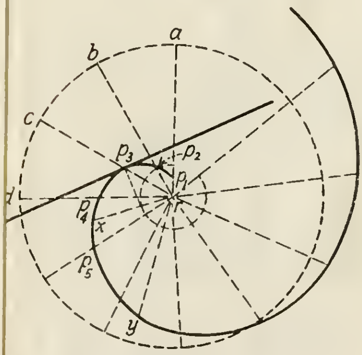

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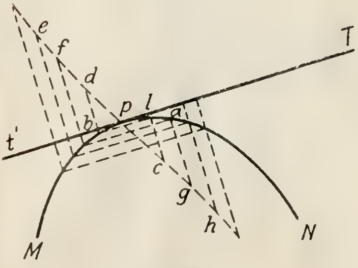
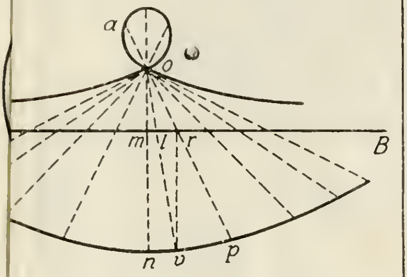


Fig. 58

Figs. 48 to 58



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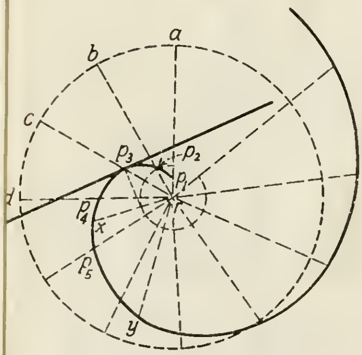

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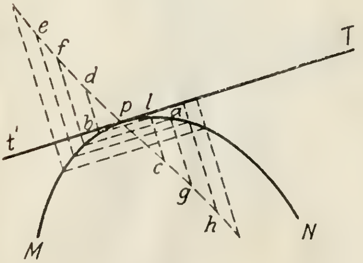

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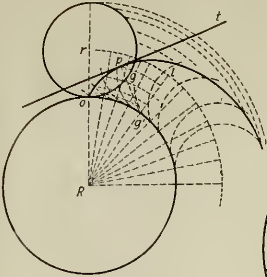

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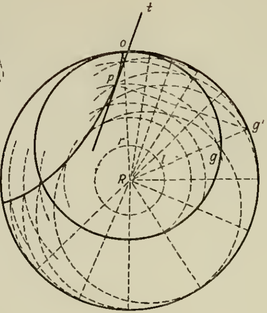

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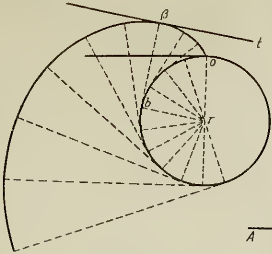

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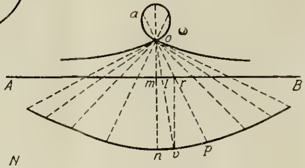

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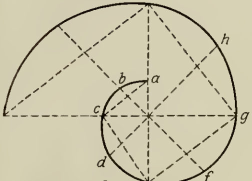

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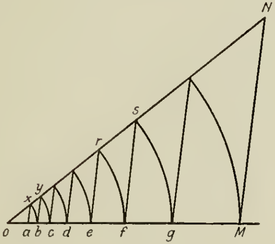

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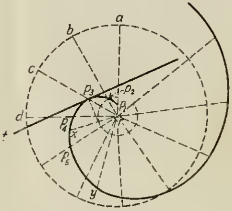

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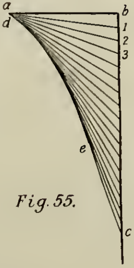

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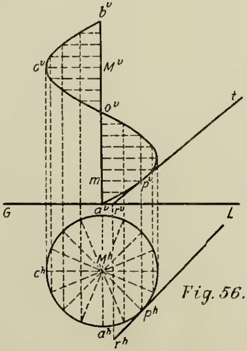

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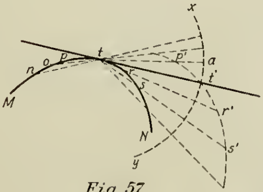

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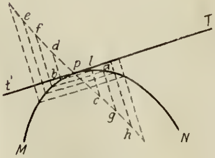

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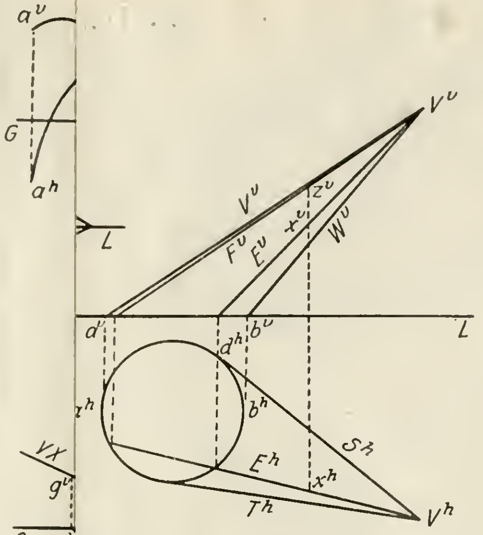

a^v
G
a^h
L
V^v
V^v
z^v
F^v
E^v
x^v
W^v
d'
d^h
b^v
L
a^h
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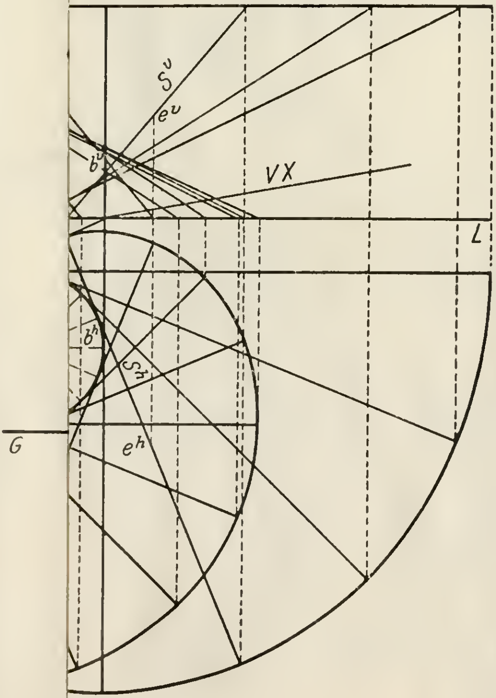
S^h
VX
E^h
x^h
g^v
T^h
V^h
G
g^h

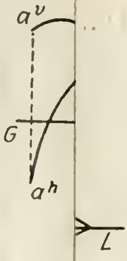
Fig. 62.

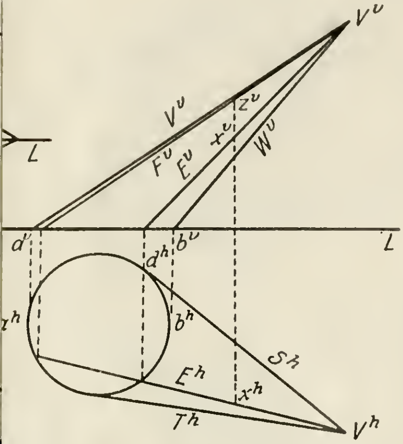


S^v
e^v
b^v
VX
L
b^h
S^h
G
e^h

5.

Figs. 59 to 66





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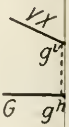

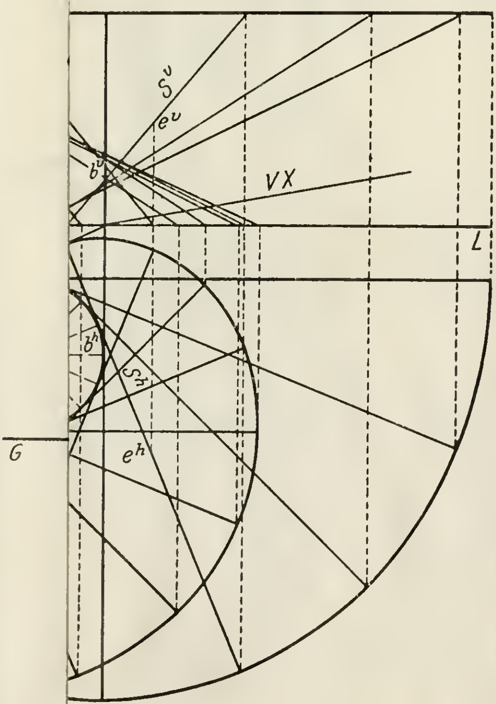

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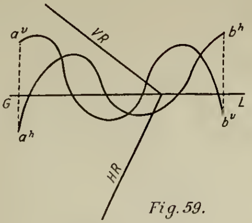

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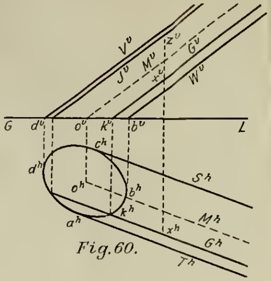

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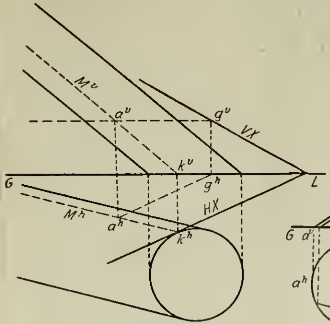

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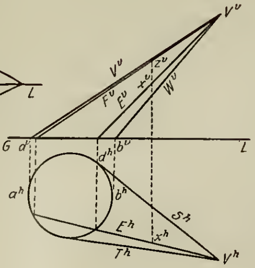

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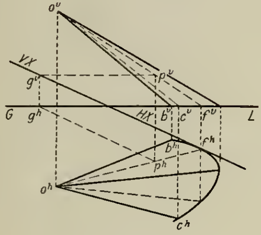

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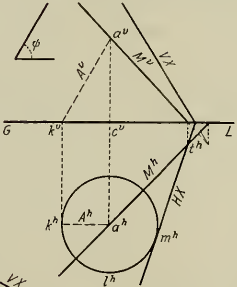

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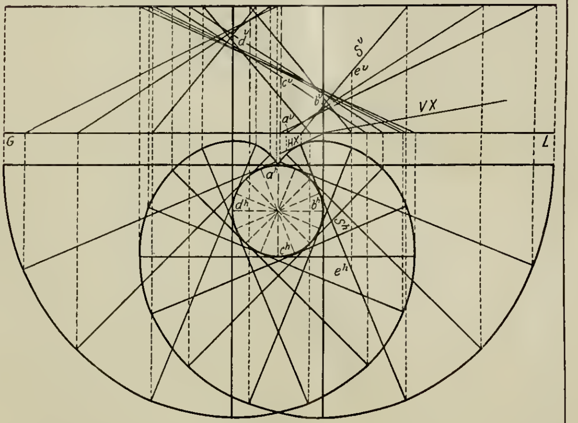

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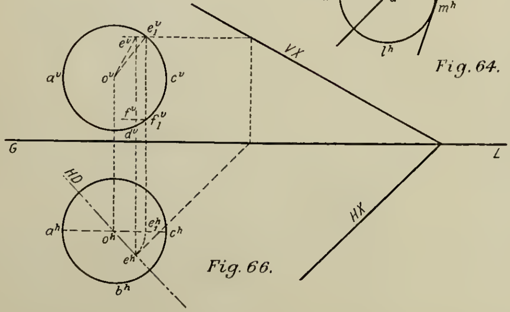
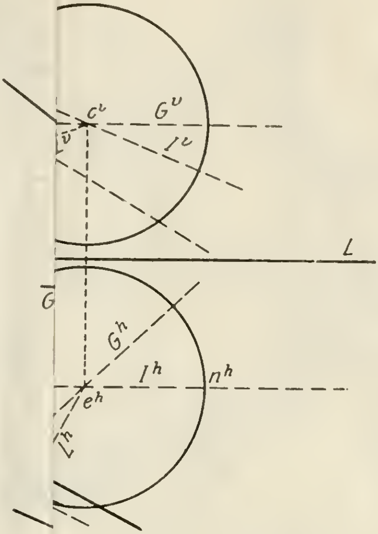


Fig. 66.

Figs. 67 to 70a



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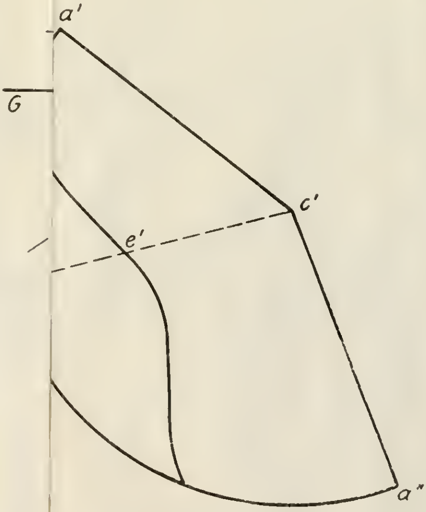
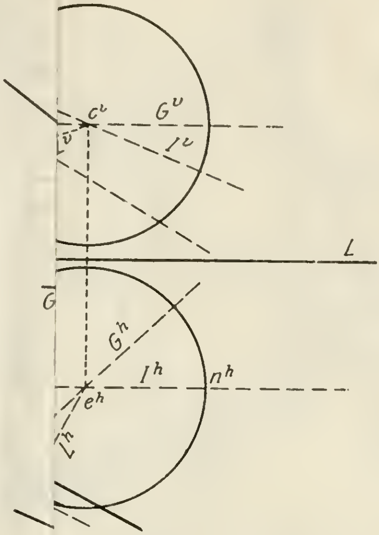


Fig. 70 a.

Figs. 67 to 70ª



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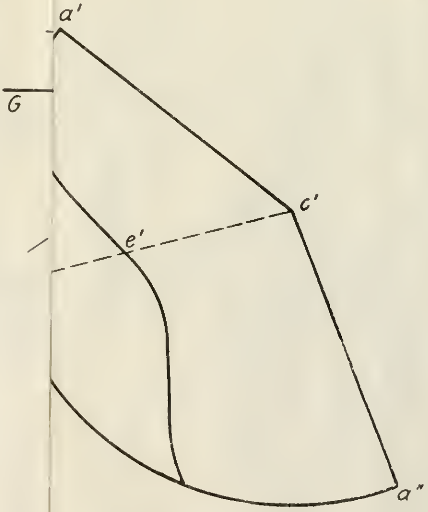

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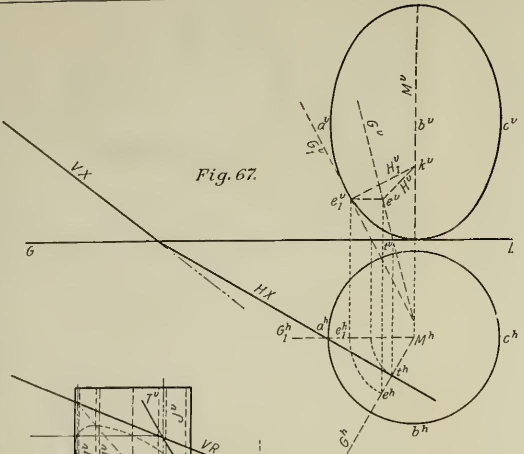

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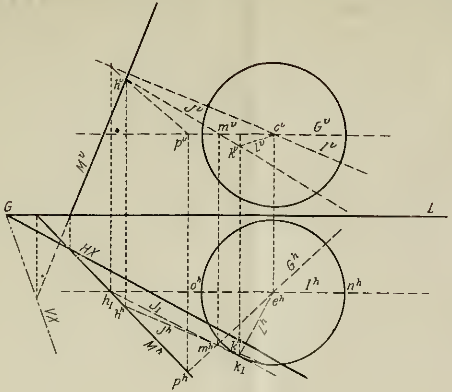

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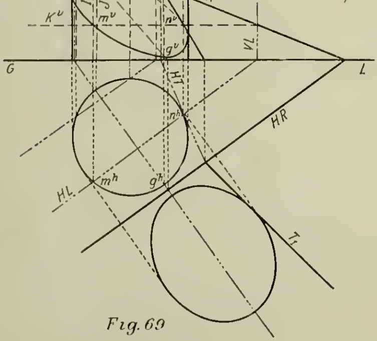

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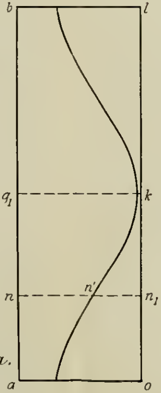

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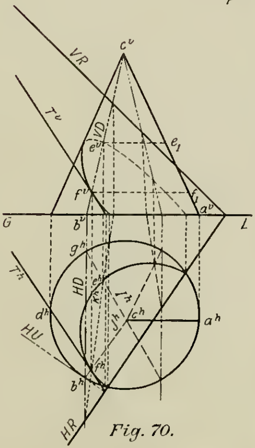

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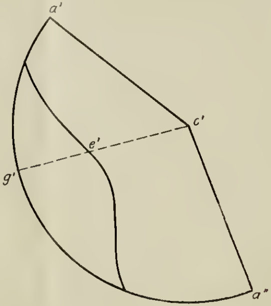
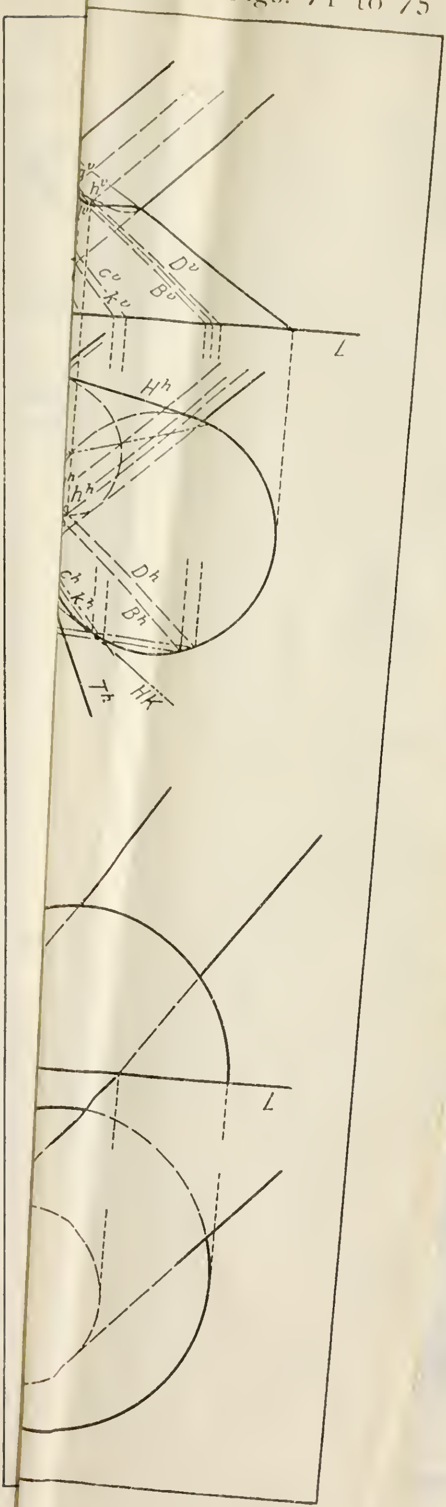
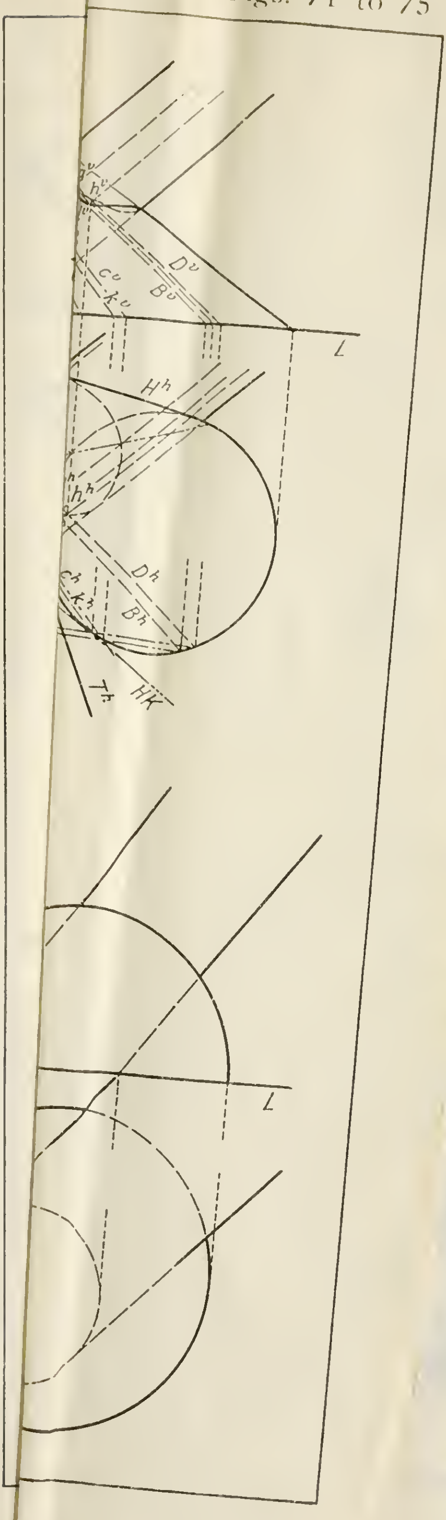


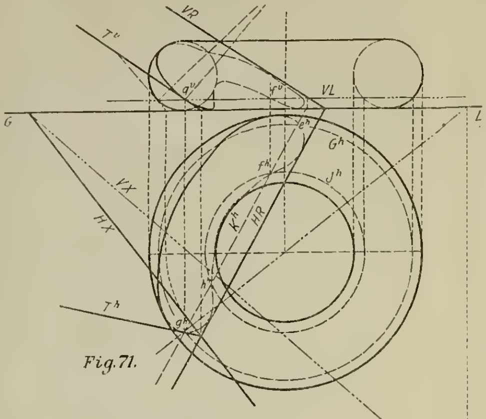
Fig. 70 a.

Figs. 71 to 75



Figs. 71 to 75





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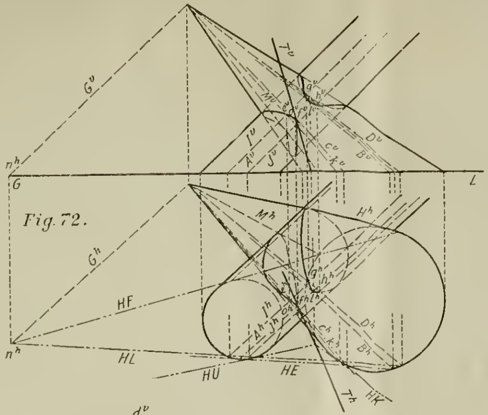

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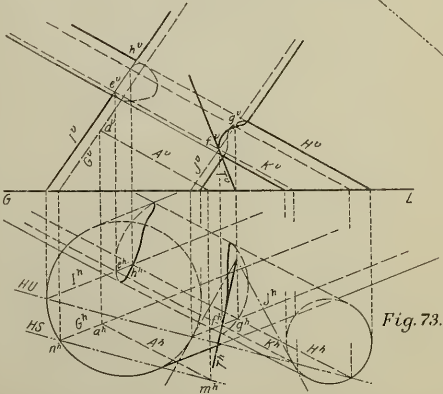

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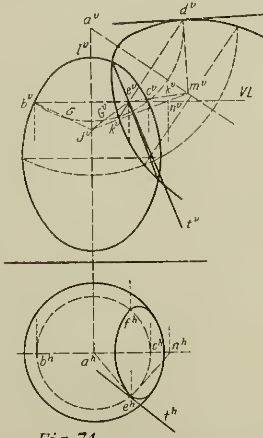

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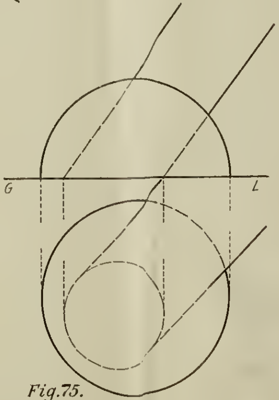
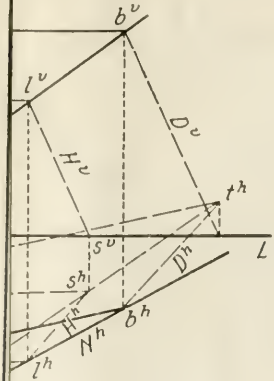


Fig. 75.

Figs. 76 to 82



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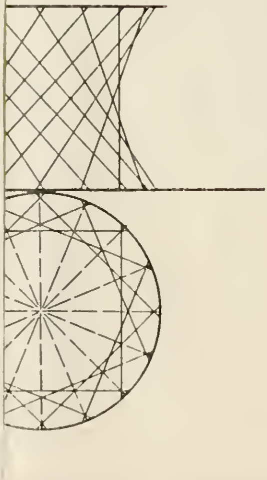
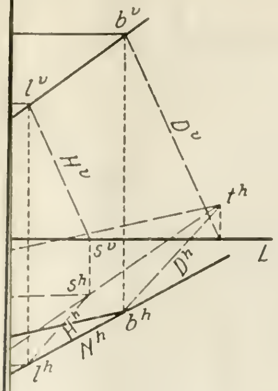


Figs. 76 to 82



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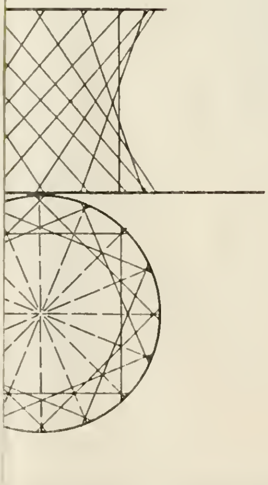

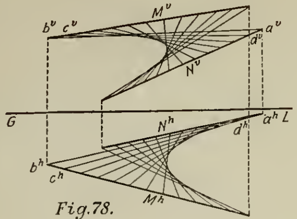

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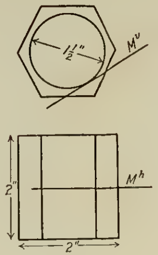

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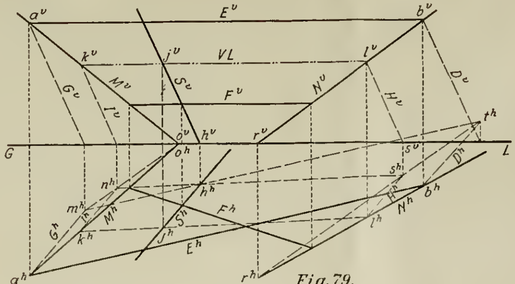

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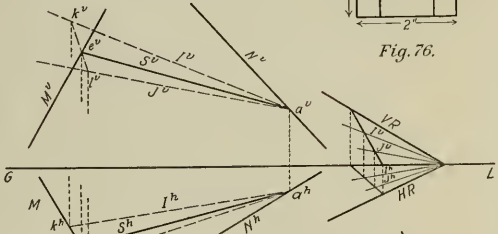

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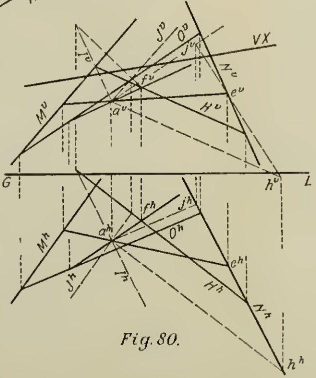

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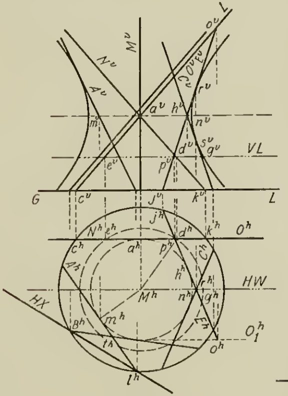
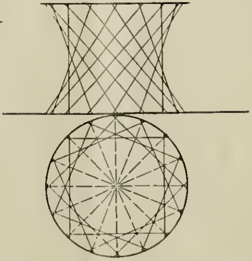
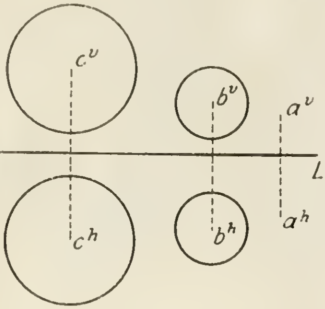


Fig. 81.

Fig. 82.



Figs. 83 to 91



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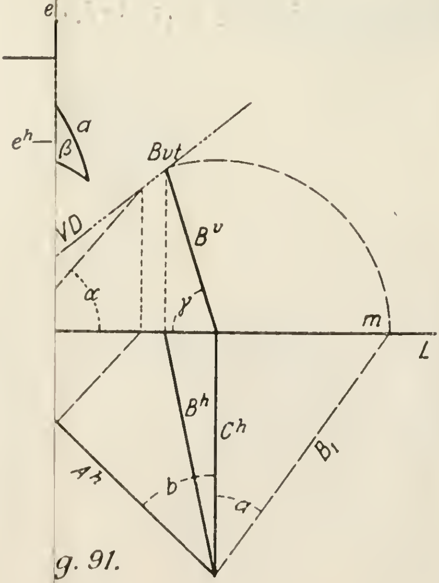
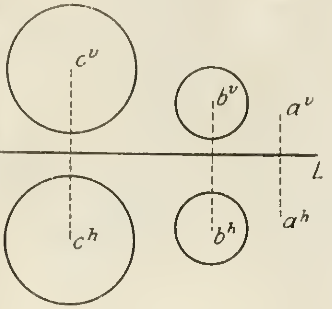


g. 91.

Figs. 83 to 91



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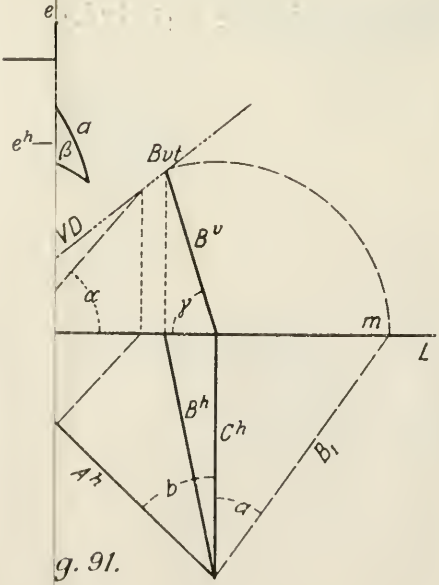

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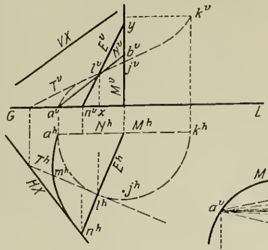

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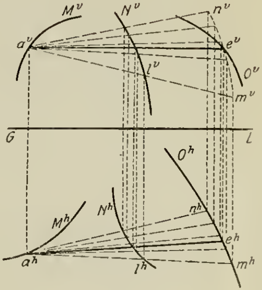

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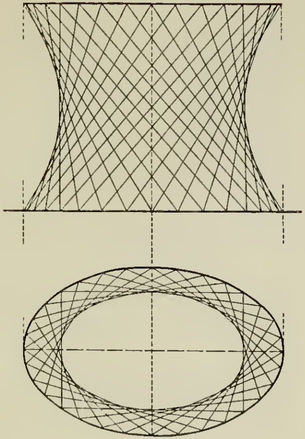

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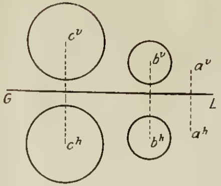

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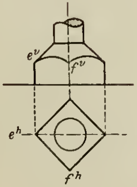

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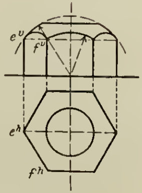

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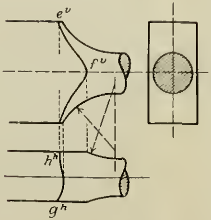

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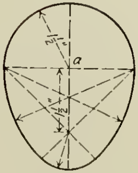

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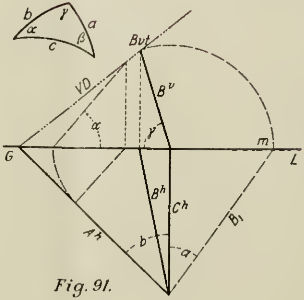

Fig. 91.

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